**PROCEEDINGS**

OF THE INSTITUTION OF ELECTRICAL ENGINEERS

**Control & Science**

# Self-tuning controller

D. W. Clarke, M.A., D.Phil., and P. J. Gawthrop, B.A.

*Indexing terms: Control-system synthesis, Controller, Optimal control, Tuning*


## Abstract

A strategy for the design of self-tuning controllers of systems with constant but unknown parameters is presented. A cost function which incorporates system input, output and set-point variations is selected, and a control law for a known system is derived. This control law is shown to comprise a least-squares predictor of a function related to the cost function, and the control input is chosen to make the prediction zero. The parameters of the control law for the unknown system are estimated using a recursive-least-squares algorithm, and the optimal parameters are shown to be a fixed point of the algorithm. Whilst retaining their computational simplicity, the proposed method has several advantages over self-tuning-regulator strategies which attempt to minimise the output variance alone: weighting of control is allowed for; set-point variation may be optimally followed; there is no requirement to choose a system-related parameter to ensure convergence; and, for stable but nonminimum phase systems, there is no need to employ time-consuming methods, such as the solution of a Riccati equation. Several simulated examples are used to demonstrate the potential of the method.


## List of principal symbols

$A(z^{-1})$ = polynomial of order $n$ corresponding to a system output; $a_0 = 1$
$B(z^{-1})$ = polynomial of order $n$ corresponding to a system input
$C(z^{-1})$ = polynomial of order $n$ corresponding to an uncorrelated random sequence; $c_0 = 1$
$d$ = constant output level
$E\{.\}$ = expectation operator
$E(z^{-1}), F(z^{-1}), G(z^{-1}), H(z^{-1})$ = general polynomials
$e_{t+k}$ = $k$-step-ahead prediction error of the output
$I, J$ = cost function
$I$ = the identity matrix
$k$ = system time delay
$K_t$ = updating gain matrix in the recursive-least-squares estimator
$n$ = system order
$P(z^{-1}), Q(z^{-1}), R(z^{-1})$ = costing polynomials for the system output, input and set point, respectively
$P_t$ = matrix proportional to the covariance of the estimated parameter
$t$ = time in sample instants (integer)
$u_t, w_t, y_t$ = system input, set point, and output, respectively, at time $t$
$x_t$ = vector containing measured data
$z$ = forward-shift operator: $z^k u_t = u_{t+k}$
$\alpha$ = asymptotic sample length
$\beta$ = forgetting factor
$\delta$ = constant related to $d$
$\epsilon_{t+k}$ = $k$-step-ahead prediction error of the generalised output function
$\lambda$ = control weighting
$\phi_t$ = generalised output function
$\boldsymbol{\theta}$ = vector of parameters
$\xi_t$ = uncorrelated zero-mean random sequence

*Paper 7492 C, first received 7th March and in revised form 29th May 1975*

*Dr. Clarke and Mr. Gawthrop are with the University of Oxford, Department of Engineering Science, Parks Road, Oxford OX1 3PJ, England*

A polynomial of order $r$ in the backward-shift operator $z^{-1}$ is denoted $P(z^{-1}) = p_0 + p_1 z^{-1} + .... + p_r z^{-r}$. After definition, and where convenient, the polynomial is written simply as $P$. The estimate of a parameter $\gamma$ is denoted by $\hat{\gamma}$.

## 1 Introduction

The optimal control of systems with unknown parameters subjected to random disturbances is of considerable theoretical and practical interest. Although this problem is soluble in principle, the excessive computations required by the resultant control laws preclude their use in practice, so it is useful to consider suboptimal algorithms which, however, may be more easily realised. An important class of methods, called 'self-tuning regulators', have been developed by Åström and Wittenmark[1,2] for the control of systems with constant, but unknown, parameters. The regulator is based on a recursive-least-squares estimator of the parameters of a feedback control law, followed by the use of the estimated parameters in the control law itself. Convergence conditions[3,4] have shown that, asymptotically, the self-tuning regulators approach the steady-state optimal regulators for a good many systems and successful practical implementations have been reported.[5] The control calculations are trivial, as the feedback consists of a pulse-transfer function; the recursive estimator is more burdensome, but not excessively so. Hence the self-tuning regulator is suitable for operation using small online computers, or even microprocessors.

As its name implies, the self-tuning regulator attempts to minimise the fluctuations of the system's output when the loop is randomly disturbed, but it makes no attempt to ensure that set points are followed optimally, nor does it try to penalise excessive control action. Work has been reported[6] which extends the regulator to include a constant set point, or which suboptimally inserts an integration in the loop,[2] but neither method is particularly appropriate, as the first ignores set-point changes, and the second will make the regulator converge to different values according to the pattern of set-point changes. Lack of costing of control is a further handicap to the basic self-tuning regulator, as there is then no way of modifying its asymptotic behaviour beyond changing the sample interval and restarting the algorithm. Moreover, to control nonminimum-phase systems in a stable manner, the simple method must be discarded, and at each sample instant the computationally more difficult online solution of a Riccati equation[2] must be used, or a polynomial identity must be resolved;[7] these complexities can be avoided if control costing is involved.[9] Identifiability conditions for the feedback parameters show that one system-related parameter must be chosen *a priori* to ensure proper operation; this detract slightly from the 'self-tuning' philosophy, and incorrect choice may lead to instability. All these problems can be overcome in a number of cases by the method described below, where the self-tuning principle is used to minimise a cost function which incorporates fluctuations of inputs, outputs and set points, whilst retaining the essential simplicity of the basic algorithm. The derivation, however, differs in emphasis from that of Åstrom and Wittenmark, and suggests how the self-tuning principle may be developed further.

The systems considered are single input and single output, randomly disturbed and described by the difference equation:

$$\sum_{i=0}^{n} a_i y_{t-i} = \sum_{i=0}^{n} b_i u_{t-i-k} + d + \sum_{i=0}^{n} c_i \xi_{t-i} \tag{1}$$

where $u_t$ and $y_t$ are the system's control input and output at the sample instant $t$ and are exactly measurable; $\xi_t$ is an uncorrelated random sequence of zero mean disturbing the system. The order of the system is $n$ and the time delay is $k$; $d$ is a constant reflecting the nonzero steady-state output of practical systems for a steady-zero control input. Without loss of generality, $a_0$ is taken to be 1; $c_0$ is further taken to be 1 and the variance of $\xi_t$ scaled accordingly. Eqn. 1 may be expressed more concisely in terms of the backward-shift operator $z^{-1}$ as:

$$A(z^{-1})y_t = z^{-k}B(z^{-1})u_t + d + C(z^{-1})\xi_t \tag{2}$$

the roots of $C$ lie within the unit circle.

The cost functions to be considered are of the form:

$$I = E\left\{\left(\sum_{i=0} p_i y_{t+k-i} - \sum_{i=0} r_i w_{t-i}\right)^2 + \left(\sum_{i=0} q_i' u_{t-i}\right)^2\right\}$$

or alternatively, if $P$, $Q'$ and $R$ are polynomials in the shift operator,

$$I = E\{(Py_{t+k} - Rw_t)^2 + (Q'u_t)^2\} \tag{3}$$

where $p_0$ may be taken to be 1 without loss of generality. The sequence $w_t$ is a known function of time, such as a set point or the output of some model which it is desired that the system should follow, and the expectation is conditional upon system input and output data acquired up to time $t$. The cost function includes the important cases:

$$I_1 = E\{(y_{t+k} - w_t)^2 + \lambda' u_t^2\} \tag{3a}$$

$$I_2 = E\{(y_{t+k} - w_t)^2 + \lambda'(u_t - u_{t-1})^2\} \tag{3b}$$

which have been used in previous control studies of systems using identified models.[10] The effect of using $I_1$ is to compromise between increased deviations of the system output and reduced control variations, whereas $I_2$ costs changes in control. If $w_t$ has a nonzero mean level $\bar{w}_t$, $I_1$ will not ensure that $\bar{y}_t$, the mean of $y_t$, is equal to $\bar{w}_t$ unless $\lambda'$ is zero. $I_2$ inserts an extra integrator into the loop and guarantees equality of $\bar{y}_t$ and $\bar{w}_t$ at the expense of degraded dynamic performance. The incorporation of the weighting polynomials $P$ and $R$ in $I$ is for the sake of completeness; the structure of the derived controller is unaffected, except for a possible increase in its order.

## 2 Controller design for systems with known parameters

As a preliminary to the proposed solution of the self-tuning control problem, consider the design of controllers which minimise $I$ of eqn. 3 for systems described by eqn. 2, but with known parameters. When $p_0$ is the only nonzero weighting parameter in $I$, the controller is the so-called minimum-variance regulator,[11] which effectively predicts the system output $k$ steps ahead, $y^*_{t+k|t}$, as a function of $u_t$, and sets the prediction to zero by appropriate choice of the current control $u_t$. Extensions to include control costs also predict and nullify a more general expression—this is the basis of the control derivation below.

Define

$$y^*_{t+j|t}(y_t, y_{t-1}, \ldots; u_t, u_{t-1} \ldots) \triangleq y_{t+j} - e_{t+j} \tag{4}$$

to be the optimal (least-squares) predictor of $y_{t+j}$ at time $t$, given data from samples up to and including time $t$. By using standard arguments, for example the orthogonal projection theorem, $e_{t+j}$ can be shown to be uncorrelated with $y_{t-i}, u_{t-i}$ for $i \geqslant 0$, and hence with $y$ itself. It is further shown in Appendix 8 that $y_{t+j|t}$ can be expressed recursively in closed form as:

$$Cy^*_{t+j|t} = F_j'(z^{-1})y_t + G_j'(z^{-1})u_{t+j-k} + \gamma_j d, \text{ for } j \leqslant k,$$

where $F_j'$, $G_j'$ and $\gamma_j$ are derived from the polynomial identities:

$$C = E_jA + z^{-j}F_j', \text{where } E_j(z^{-1}) \text{ is of order } j-1;$$

$$G_j' = E_jB_j$$

$$\gamma_j = E_j(1)$$

We may set, trivially, $y^*_{t+j|t} = y_{t+j}$ for $j \leqslant 0$, which may be included in the above recursion by putting $F_j' = z^jC$ and $E_j = 0$ for $j \leqslant 0$.

Now, substituting eqn. 4 into eqn. 3:

$$I = E\{(P[y^*_{t+k|t} + e_{t+k}] - Rw_t)^2 + (Q'u_t)^2\}$$

where $Pe_{t+k}$ is uncorrelated with $u_{t-i}, w_{t-i}, y_{t-i}$ for $i \geqslant 0$. Therefore, letting $\sigma^2 = E\{(Pe_{t+k})^2\}$, the cost function becomes:

$$I = (Py^*_{t+k|t} - Rw_t)^2 + (Q'u_t)^2 + \sigma^2$$

which is minimised by choosing $u_t$ such that:

$$\frac{\partial I}{\partial u_t} = 2(Py^*_{t+k|t} - Rw_t)G_k'(0) + 2q_0'Q'u_t = 0$$

Using the fact that $G_k'(0) = b_0$, the control law is given by:

$$Py^*_{t+k|t} + \frac{q_0'Q'}{b_0}u_t - Rw_t = 0 \tag{5}$$

Defining a new costing polynomial $Q(z^{-1})$ so that $Q = \dfrac{q_0'Q'}{b_0}$, and a function $\phi^*$ such that

$$\phi^*_{t+k|t} = Py^*_{t+k|t} + Qu_t - Rw_t \tag{6}$$

the control law given by eqn. 5 is written concisely as:

$$\phi^*_{t+k|t} = 0 \tag{7}$$

Furthermore, if $\phi_{t+k}$ is defined as:

$$\phi_{t+k} = Py_{t+k} + Qu_t - Rw_t$$

and recalling that $y_{t+j} = y^*_{t+j|t} + e_{t+j}$, where $e_{t+j}$ is uncorrelated with $y^*_{t+j|t}$, it follows that:

$$\phi_{t+k} = \phi^*_{t+k|t} + \epsilon_{t+k}$$

where

$$\epsilon_{t+k} = \sum_{i=0}^{k-1} p_i e_{t+k-i}$$

is uncorrelated with $\phi^*$, and that the function $\phi^*$ is the least-squares optimal predictor of $\phi$. Moreover, defining a cost function $J$ to be $E\{\phi^2_{t+k}\}$, then:

$$J = (\phi^*_{t+k|t})^2 + \sigma^2$$

and minimising $I$ or $J$ generates the same control law.

Now, substitute for $y^*_{t+j|t}$ in eqn. 6 to get:

$$\phi^*_{t+k|t} = \sum_{j \geqslant 0} \frac{p_j}{C}[F'_{k-j}y_t + G'_{k-j}u_{t-j} + \gamma_{k-j}d] + Qu_t - Rw_t,$$

and define new polynomials:

$$F(z^{-1}) = \sum p_j F'_{k-j}$$

$$G(z^{-1}) = \sum p_j z^{-j} G'_{k-j} + CQ$$

$$H(z^{-1}) = -CR$$

and a constant: $\delta = \Sigma p_j \gamma_{k-j} d$. It follows that $\phi^*$ may itself be expressed recursively as:

$$C\phi^*_{t+k|t} = Fy_t + Gu_t + Hw_t + \delta \tag{8}$$

and as the control sets $\phi^*$ to zero at each stage, it is given by

$$Fy_t + Gu_t + Hw_t + \delta = 0 \tag{9}$$

This control algorithm is similar to that of the basic minimum variance control in that it minimises $E\{\phi_t^2\}$ by setting the predicted future value of $\phi$ equal to zero at each step. Just as the minimum variance control law also minimises the infinite stage variance of the output $y_t$, and not simply the conditional cost function, so the control law derived above minimises the infinite stage variance of the generalised output $\phi_t$. This is useful when considering its behaviour in controlling nonminimum phase systems. Note also that, as $\phi^*$ is set to zero at each stage:

$$\phi_t = \phi^*_{t|t-k} + \epsilon_t = \epsilon_t$$

Hence

$$E\{\phi_t \phi_{t+s}\} = 0, \quad |s| \geqslant k$$

$$E\{\phi_t y_{t-s}\} = 0 = E\{\phi_t u_{t-s}\}, \quad s \geqslant k$$

and

$$E\{\phi_t w_{t-s}\} = 0, \quad \text{all } s$$

These correlation functions may be used empirically to verify correct controller action. The structure of the control law is shown in Fig. 1: it contains forward, feedback and precompensator elements which are all simple to realise digitally. The constant $\delta$ ensures a proper output level without the use of an integrator in the forward path. In practice, $d$ may drift, so an integrator would be required in the loop, unless $\delta$ were updated continuously to compensate.

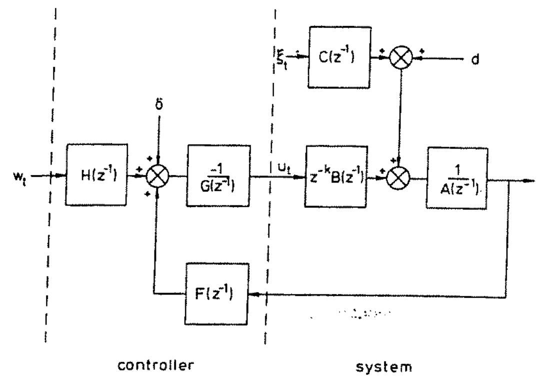


**Fig. 1**
*Structure of the optimal controller*

The stability of the feedback system may be shown to depend on the location of the roots of the polynomial equation:

$$PB + QA = 0 \tag{10}$$

For example, if $I_1$ is chosen as the cost function, then $P = 1$, so if $\lambda' = b_0\lambda$ then $Q = q_0'Q'/b_0 = \lambda$ and stability is determined by roots of

$$B + \lambda A = 0$$

For $\lambda = 0$, we obtain the minimum-variance regulator and the roots depend entirely on the factors of $B$; hence, for nonminimum phase systems, the closed loop is unstable. However, if $\lambda \neq 0$, the roots now depend on $A$ also; if the system is open-loop stable these will lie within the unit circle and the system will be closed-loop stable provided $\lambda$ is large enough. Alternatively, unstable, minimum phase systems are stabilised if the control weighting $\lambda$ is small enough.

It is interesting to consider the sensitivity of the control behaviour to the use of estimated rather than true parameters. A full analysis involves extensive algebra and is outside the scope of this paper; here we restrict ourselves to the special case where $k = 1 = P, d = 0 = R$, $Q = \lambda$, giving:

$$\phi_{t+1} = y_{t+1} + \lambda u_t, \quad \text{so}$$

$$C\phi_{t+1} = Fy_t + (B + \lambda C)u_t + C\xi_{t+1}$$

The control law here is:

$$\hat{F}y_t + (\hat{B} + \lambda\hat{C})u_t = 0$$

where $\hat{F}$ denotes an estimate of the polynomial $F$ etc. Then, using eqn. 2 we get a recursive relation for $\phi_t$:

$$\phi_{t+1} = \frac{\{F(\hat{B} + \lambda\hat{C}) - \hat{F}(B + \lambda C)\}(B\phi_t + \lambda C\xi_t)}{C(B + \lambda A)(\hat{B} + \lambda\hat{C})} + \xi_{t+1} \tag{11}$$

Consider first the minimum-variance case where $\lambda = 0$: eqn. 11 reduces to

$$y_{t+1} = \frac{F\hat{B} - \hat{F}B}{C\hat{B}}y_t + \xi_{t+1}$$

Now define $\tilde{B}(z^{-1}) = B - \hat{B}$ to be the error in the $B$ polynomial, and suppose $\hat{F} = F$, then:

$$y_t = \frac{\xi_t}{1 + T(z^{-1})}, \quad \text{where } T = \frac{\tilde{B}(C - A)}{\hat{B}C}, \text{ as } F = z(C - A)$$

This means that $y_t$ may be considered to be the output of a closed-loop system with $\xi_t$ as input, and for which the open-loop transfer function is $T(z^{-1})$. Hence the sensitivity may be analysed by using classical methods, such as the Nyquist plot or the root locus, and situations may readily be constructed for which the 'closed loop' is unstable and the control is infinitely sensitive to the given $\tilde{B}$.

The corresponding analysis for general $\lambda$ in eqn. 11 is difficult, but if we assume that $\phi_t = \xi_t$ (which it would be in the case of no parameter error), the corresponding closed-loop 'sensitivity' system is approximately:

$$\phi_t = \frac{\xi_t}{1 + T(z^{-1})}, \quad \text{where } T = \frac{\tilde{B}(C - A)}{C(B + \lambda A)}$$

If the original system, eqn. 2, is open-loop stable, then for large enough $\lambda$ we can make this 'sensitivity' system stable in cases where the simpler version is not, and the corresponding control becomes relatively insensitive to parameter error; this has been confirmed in some simulated examples in Reference 8.

## 3 Self-tuning controller

In the previous Section, it was shown that the control law for a system with known parameters reduced to setting the optimal prediction $\phi^*_{t+k|t}$ of a function $\phi_{t+k}$, closely related to the chosen cost function, to zero at each stage. Effectively the system, eqn. 1, and the cost function, eqn. 3, are combined to form an equivalent system

$$\begin{aligned} C\phi^*_{t+k|t} &= Fy_t + Gu_t + Hw_t + \delta \\ \phi_{t+k} &= \phi^*_{t+k|t} + \epsilon_{t+k} \end{aligned} \tag{12}$$

and with the control setting $\phi^*$ to zero the closed-loop behaviour is such that $\phi_{t+k}$ equals $\epsilon_{t+k}$. In this Section, the problem of the control of systems described by eqn. 1, but with unknown parameters, is considered, in which simplicity of the algorithm is preferred to a strictly optimal solution. The cost function is again given by eqn. 3, so eqn. 12 then describes the equivalent system, but now parameters of $F, G, H$ and $\delta$ are unknown.

If $C = 1$, eqns. 12 can be written in the form:

$$\phi_{t+k} = x_t^T\boldsymbol{\theta} + \epsilon_{t+k} \tag{13}$$

where $x_t$ is a column vector containing known functions of time:

$$x_t^T = \{y_t, y_{t-1} \ldots ; u_t, u_{t-1} \ldots ; w_t, w_{t-1} \quad ; 1\}$$

and $\boldsymbol{\theta}$ is the vector of unknown parameters:

$$\boldsymbol{\theta}^T = \{f_0, f_1 \ldots ; g_0, g_1 \ldots ; h_0, h_1 \ldots ; \delta\}$$

Now, as the components of $x_t$ are uncorrelated with $\epsilon_{t+k}$, it is well known that the best (minimum variance and unbiased) linear estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is given using a least-squares algorithm,[12] which is most usefully expressed in this context in its recursive form. At time $t$, let the vector of parameter estimates previously available be $\hat{\boldsymbol{\theta}}_{t-1}$. As a new value of $y_t$ is now acquired $\phi_t$ may be constructed, so a new estimate $\hat{\boldsymbol{\theta}}_t$ of the parameters may be generated:

$$\left.\begin{aligned} \hat{\boldsymbol{\theta}}_t &= \hat{\boldsymbol{\theta}}_{t-1} + K_t(\phi_t - x_{t-k}\hat{\boldsymbol{\theta}}_{t-1}) \\ K_t &= P_t x_{t-k}(1 + x_{t-k}^T P_t x_{t-k})^{-1} \\ P_t &= P_{t-1} - K_{t-1}(1 + x_{t-k}^T P_t x_{t-k}K_{t-1}^T \end{aligned}\right\} \tag{14}$$

where $P_t$ is a matrix proportional to the covariance of the estimated parameters. The simplest control takes no account of any interactions that may exist between identification and control, but merely assumes that the estimated parameters are in fact the true ones, and hence chooses $u_t$ such that:

$$\hat{\phi}^*_{t+k|t} = \hat{F}y_t + \hat{G}u_t + \hat{H}w_t + \hat{\delta} = x_t^T\hat{\boldsymbol{\theta}}_t = 0 \tag{15}$$

where $\hat{F}, \hat{G}, \hat{H}, \hat{\delta}$ are the estimated values of $F, G, H$ and $\delta$.

Now, in general, $C$ is a polynomial of order $n$, so eqn. 13 becomes:

$$\phi_{t+k} = x_t^T\boldsymbol{\theta} + \epsilon_{t+k} + (1 - C)\phi^*_{t+k|t} \tag{16}$$

In this case, $\phi^*$ is correlated with $x$, so the use of a least-squares estimator alone would produce estimates which are no longer unbiased. But the estimator is cascaded with a control law which sets $\hat{\phi}^*$ to zero. Hence, assuming $\hat{\boldsymbol{\theta}} \to \boldsymbol{\theta}$, the control law, eqn. 15, tends to the optimal control law $\phi^* = 0$, and the offending terms in eqn. 16 will vanish. Therefore $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$ is a fixed point of the algorithm; its stability is now briefly discussed.

In References 3 and 4, Ljung has shown, for the self-tuning regulator, that if the input and output of the system are bounded, which naturally requires stable closed-loop control, stability of the fixed point of the algorithm, $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, is related to the stability of an associated ordinary differential equation. The conditions which he derives for obtaining systems which produce unstable fixed points, and where their parameter estimates oscillate about the point, are closely related to the parameter sensitivity criteria for the minimum-variance control law discussed earlier. This indicates that a small error in the parameter vector is rapidly propagated into poor control in these cases, and this can further bias future parameter estimates. However, the previous discussion also demonstrated that this excessive sensitivity may be reduced by appropriate choice of $\lambda$, so convergence is possible in a wider range of circumstances.

The useful asymptotic behaviour of combined least-squares estimator and least-squares predictor control law is due to the similarity of the cost functions minimised by each. The estimator generates $\boldsymbol{\theta}_t$ to minimise the function

$$I_E(\hat{\boldsymbol{\theta}}_t) = \frac{1}{t}\sum_{s=1}^{t}(\phi_s - x_{s-k}^T\hat{\boldsymbol{\theta}}_t)^2$$

whereas the predictor uses a value $\boldsymbol{\theta}^*$ which minimises:

$$I_p(\boldsymbol{\theta}^*) = E\{(\phi_{t+k} - x_t^T\boldsymbol{\theta}^*)^2\}$$

assuming that the system parameters are known. Although the sequence $x_t$ is not stationary even if $w_t$ is, for $\hat{\boldsymbol{\theta}}_t$ varies with time, so that the ergodic theorem cannot be invoked to directly equate $I_E$ and $I_p$, for large value of $t$, the estimate $\hat{\boldsymbol{\theta}}_t$ is only varying slowly, and the two functions become approximately equivalent. Suppose that the estimation is poor in the sense that $I_E$ is relatively unaffected by a change in the estimates in some direction in parameter space. Then the predictor, and hence the control law, will be insensitive to a similar deviation; so good control will be achievable even if parameter estimates vary.

As pointed out in Reference 1, $I_E$ and $I_p$ are constant in one direction in parameter space, that of all vectors of the form $\mu\boldsymbol{\theta}$. In principle, the above indicates that this is unimportant as far as the control is concerned, but in practice numerical problems would arise if $\mu$ were very large or very small. Following Reference 1, this is circumvented by choosing a value for one component, the $i$th say, to be $\hat{\theta}_i$, thus fixing $\mu$ to be $\hat{\theta}_i/\theta_i$. For finite $\mu$, a nonzero element of $\boldsymbol{\theta}$ must of course be chosen; if no set-point terms are present, and so $H(z^{-1})$ is not included, one parameter that can never be zero, provided $k$ has been properly chosen, is $g_0$ the leading coefficient of $G(z^{-1})$, which depends on unknown system parameters. It has been shown using a particular example[1] that if this parameter is taken to be much smaller than its actual value, the estimates $\hat{\boldsymbol{\theta}}_t$ may diverge, and if taken too large convergence tends to be slow. However, if $w_t$ is nonzero, the corresponding parameters of $H(z^{-1})$ are included in the vector $\boldsymbol{\theta}$. The leading coefficient $h_0$ is independent of the system parameters, and hence is known *a priori*. It is, therefore, the most convenient parameter to choose in that these related convergence problems do not arise.

If $C \neq 1$, convergence of the algorithm is hindered by the inclusion of early data $\phi_t$ which, as the control law is away from the optimum, contains components due to nonzero $\phi^*_{t/t-k}$. The standard recursive-least-squares estimator is derived ignoring these terms, so the norms of $P_t$ and hence $K_t$ of eqn. 14 may approach zero more rapidly than $\phi^*$, and the estimates $\boldsymbol{\theta}_t$ may only be changing slowly, even though they are relatively far from the true parameters. This effect can be overcome by slightly modifying the estimator so it minimises the function

$$I_E(\hat{\boldsymbol{\theta}}_t) = \frac{1}{t}\sum_{s=1}^{t}\beta^{t-s}(\phi_s - x_{s-k}^T\hat{\boldsymbol{\theta}}_t)^2$$

where $\beta$ is the 'forgetting factor', and is a constant less than unity. The function of $\beta$ in $I$ is to weigh out past data; a useful figure here is the 'asymptotic sample length' $\alpha$, given by

$$\alpha = \sum_{s=0}^{\infty}\beta^s = \frac{1}{1-\beta}$$

which shows that $\beta$ must in fact be close to unity so that fluctuations in the data are not excessively reflected in the estimates.

## 4 Simulation of the self-tuning controller

The simulated examples discussed below illustrate the behaviour of the self-tuning controller and compare its properties with those of algorithms presented elsewhere.[2,7] In each example, both the system and the controller were simulated digitally and the time delay and the polynomial orders appropriate for the controller were correctly chosen. The initial parameters were usually taken to be zero (except for $g_0$ to avoid control saturation on start up), the initial matrix $P_0$ to be $100\,I$, and asymptotic sample lengths to be in the range 200–500. After a typical run of 1000 sample instants, a standard series of graphs were plotted, the correlation functions of the data over the last half of the run were computed and the performance in comparison with the optimal controller for the system with known parameters was evaluated.

The Figures associated with each example contain the following graphs: $w$, the set-point, or the desired output of the system; $\phi$, the generalised function, the variance of which the controller attempts to minimise; $u$, the controller output, being the input to the system; estimated parameters, the estimated controller parameters, the broken lines being the parameters of the optimal controller for known system polynomials; c.c.f., the normalised correlation functions over the last 501 points for shifts $s$ in the range $[-10, 10]$. These are:

$$\frac{1}{501}\sum_{t=500}^{1000}\phi_t\phi_{t-s}$$

which should be approximately zero for $|s| \geqslant k$

$$\frac{1}{501}\sum_{t=500}^{1000}\phi_t y_{t-s}, \quad \frac{1}{501}\sum_{t=500}^{1000}\phi_t u_{t-s}$$

which should both be approximately zero for $s \geqslant k$

$$\frac{1}{501}\sum_{t=500}^{1000}\phi_t w_{t-s}$$

which should be approximately zero for all $s$; Cumulative excess loss, if $\phi'_t$ is the generalised function for the optimal controller at time $t$ operating with the same system and disturbance sequence, then this is the function

$$\sum_{s=1}^{t}\{(\phi_s)^2 - (\phi'_s)^2\}.$$

If the self-tuning controller converges to the optimal controller, this function will clearly tend to a constant value.

### 4.1 Example 1

The system involved is stable, minimum phase, and 2nd-order

$$y_t = 1{\cdot}5y_{t-1} - 0{\cdot}7y_{t-2} + u_{t-1} + 0{\cdot}5u_{t-2} + \xi_t - 0{\cdot}5\xi_{t-1} + 0{\cdot}4$$

and the cost-function to be minimised is

$$I = E\{(y_{t+1} - w_t)^2 + 0{\cdot}5u_t^2 \mid y_t, u_t \ldots\}$$

giving a generalised function

$$\phi_t = y_t - w_{t-1} + 0{\cdot}5u_t$$

As a numerical example of the use of the theory in Section 2, we shall derive the extended minimum variance control law. Here $k = 1$, so

$$C = A + z^{-1}F' \quad \text{and} \quad E = 1$$

Hence, $F'_1 = 1{\cdot}0 - 0{\cdot}7z^{-1}$; $G'_1 = EB = 1{\cdot}0 + 0{\cdot}5z^{-1}$; $\gamma = E(1) = 1{\cdot}0$. Recalling that:

$$F \triangleq \sum p_i F'_{k-i} = F'_1$$

$$G \triangleq \sum p_i z^{-i} G'_{k-i} + CQ = G'_1 + 0{\cdot}5C$$

$$H \triangleq -CR = -C$$

we have

$$C\phi^*_{t+1|t} = 1{\cdot}0y_t - 0{\cdot}7y_{t-1} + 1{\cdot}5u_t + 0{\cdot}25u_{t-1} - w_t + 0{\cdot}5w_t + 0{\cdot}4$$

Hence the optimal control law given the system parameters is:

$$u_t = (1{\cdot}0y_t - 0{\cdot}7y_{t-1} + 0{\cdot}25u_{t-1} - w_t + 0{\cdot}5w_{t-1} + 0{\cdot}4)/1{\cdot}5$$

The self-tuning controller was realised choosing the coefficient of $w_t (h_0)$ to be its known value of $-1$, and the other six parameters estimated; the control law used the estimated parameters at each sample instant as if they had the true values. The initial value of $\hat{g}_0$ was chosen to be $2{\cdot}0$, instead of the true value of $1{\cdot}5$, and the asymptotic sample length $\alpha$ was chosen to be 200. Fig. 2 shows a typical run with this system; the graphs of $y_t$ and of the excess cumulative loss both indicate that satisfactory control was achieved within 20 sample instants. The behaviour of the estimated parameters, however, show that the asymptotic sample length was too short for good estimates, but the control itself and the correlations are both good, which implies that the control strategy is not very sensitive to such variations of the estimates.

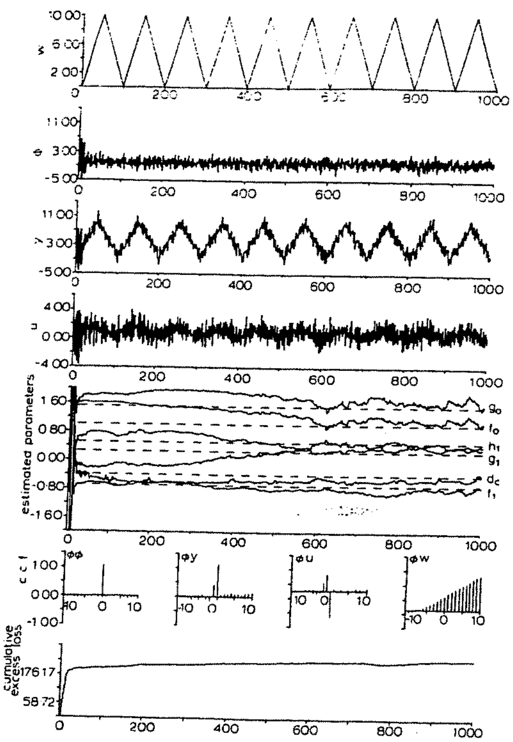


**Fig. 2**
*Controlling a 2nd-order system*

### 4.2 Example 2

In Reference 2, it is shown that the basic self-tuning regulator can to some extent accommodate set points by letting the algorithm operate on the difference between the system output and the set point, in place of the output itself, and by inserting an integrator in the closed loop. The regulator converges so as to produce the minimum error variance only if either the setpoint is constant, or if the noise is zero. If the set point varies and noise is present, the structure of the regulator does not allow convergence to the optimal control law, and so the associated cost function is not minimised. Fig. 3 shows the results of simulating such a regulator, the system being

$$y_t = 0{\cdot}9\, y_{t-1} + u_{t-1} + \xi_t + 0{\cdot}5\, \xi_{t-1}$$

To allow for comparisons with the self-tuning controller, the cost function is taken to be $E\{\phi_t^2\}$, where $\phi_t = y_t - w_{t-1}$. Note that the excess cumulative loss is continually increasing, and that the autocorrelation of $\phi_t$ is not optimal. The parameters, morever, converge to values close to those of the generalised dead-beat controller which are indicated in the Figure, as the signal-to-noise ratio is high, but they would converge to different values according to the pattern of set-point changes.

The behaviour of this adaptation of the self-tuning regulator should be compared with that of the self-tuning controller for the same system, as shown in Fig. 4. In situations where the set-point rarely changes, however, these differences would not be as marked except at the changes themselves.

### 4.3 Example 3

The system controlled is stable but nonminimum phase:

$$y_t = 0{\cdot}95 y_{t-1} + 1{\cdot}0 u_{t-2} + 2{\cdot}0 u_{t-3} + \xi_t - 0{\cdot}7\xi_{t-1}$$

This example is discussed in Reference 7 where it is shown that, although the basic self-tuning regulator is unstable, an algorithm which factorises the estimated polynomials and excludes the terms causing instability can exert satisfactory control. This system is tackled here using the self-tuning controller by recognising that a choice of $\phi_t$ in the form $\phi_t = y_t + \lambda u_{t-1}$ leads to a stable control law if $\lambda$ is greater than a certain value. Applying eqn. 10, the characteristic equation is:

$$(1 + 2{\cdot}0 z^{-1}) + \lambda(1 - 0{\cdot}95 z^{-1}) = 0$$

Hence, for $\lambda > 0{\cdot}514$, the closed-loop poles are within the unit circle and the control is stable.

The system and the controller were simulated with $\lambda$ chosen to be

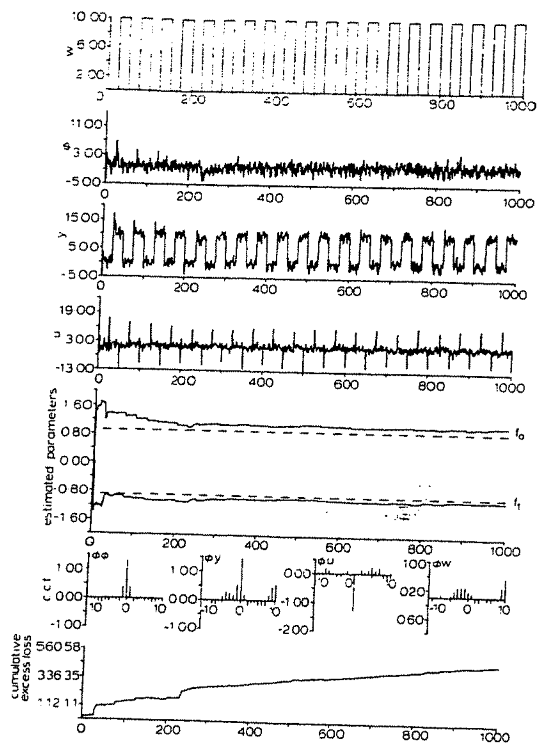


**Fig. 3**
*Example 2: Using Wittenmark's modification of the self-tuning regulator*

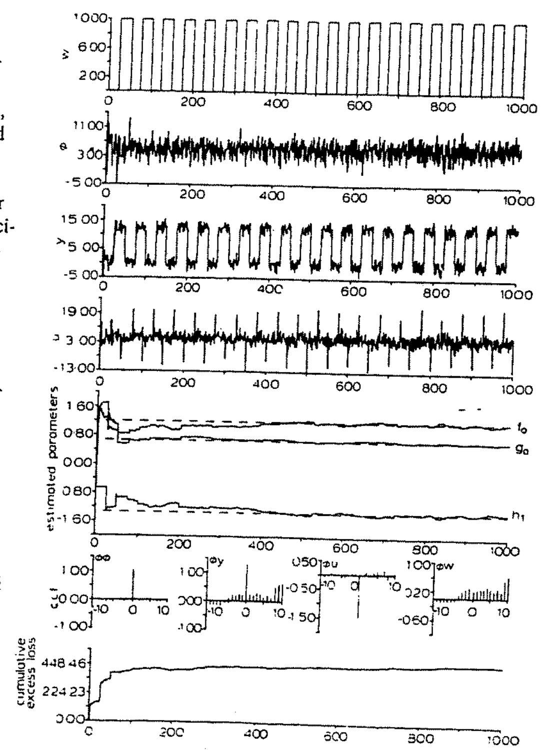


**Fig. 4**
*Example 2: the self-tuning controller with the same system as Fig. 3*

0·7 and with an asymptotic sample length of 200. The control signal $u_t$ was restricted by hard limits of ±10 to avoid initial excessive control action. Fig. 5 shows the behaviour during a typical run; after the first few sample instants the loop is clearly stable. Fig. 6 shows the excess cumulative loss of $y_t$, as well as $\phi_t$ and the autocorrelation of $y_t$; the variance of $y_t$ is not greatly above the minimum achievable for this system.

An obvious question concerns the practical choice of $\lambda$, for in this example $\lambda$ was clearly chosen with reference to the system parameters, which are in general assumed to be unknown. One method, successfully used in References 8 and 10, is to choose a large value of $\lambda$ initially to

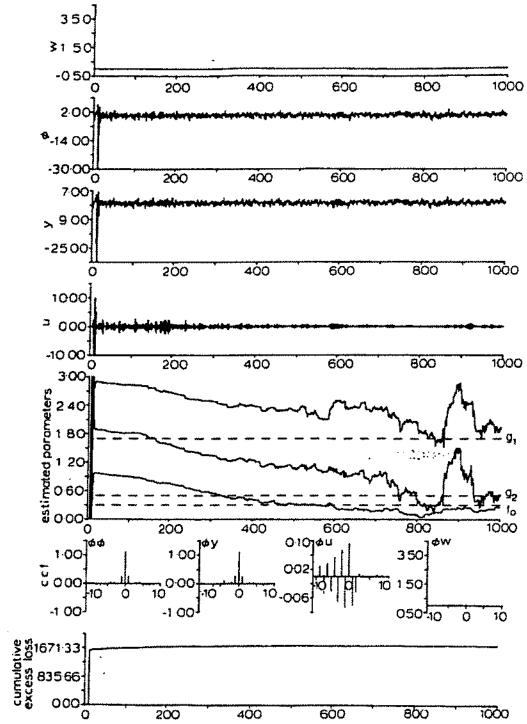


**Fig 5**
*Example 3: controlling a nonminimum-phase system*

restrict control action, and then to reduce $\lambda$ later to improve the overall performance. This strategy could also be useful even if the system is known to be minimum phase.

## 5 Conclusions

A self-tuning controller has been derived which minimises a cost-function containing terms in the system input and set point as well as in the output. The controller retains the essential simplicity of the self-tuning regulator of Åström and Wittenmark, and should be of value in industrial situations where process parameters are unknown. The

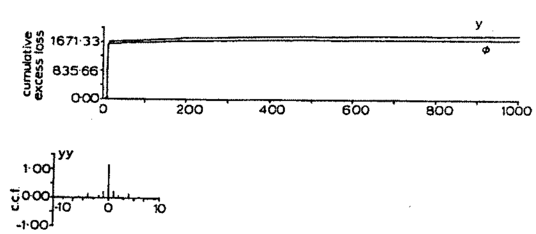


**Fig. 6**
*Example 3: variance of the output of the nonminimum-phase system*

properties of the controller have been shown by examples to be superior to those of the self-tuning regulator; many further examples have also been simulated which verify the generally good behaviour of the algorithm. It is believed that the derivation of the algorithm illuminates fundamental aspects of the controller and the approximations involved.

The controller is shown to have good asymptotic behaviour and it is hoped to report on methods for improving its transient behaviour shortly.

## 6 Acknowledgment

We would like to thank the Science Research Council for financial support of one of the authors (P. J. G.), and for support of a related project investigating the feasibility of the use of microprocessors as self-tuning regulators.

## 7 References

1 ÅSTRÖM, K.J., and WITTENMARK, B.: 'On self-tuning regulators', *Automatica*, 1973, 9, pp. 185–199
2 WITTENMARK, B.: 'A self-tuning regulator', Division of Automatic Control, Lund Institute of Technology, Report 7311, 1973
3 LJUNG, L., and WITTENMARK, B.: 'Analysis of a class of adaptive regulators'. Proceedings of the IFAC stochastic control symposium, Budapest, 1974, pp. 431–437
4 LJUNG, L., and WITTENMARK, B.: 'Asymptotic properties of self-tuning regulators'. Lund Institute of Technology, Division of Automatic Control, Report 7404, 1974
5 BORRISON, V., and SYDING, R.: 'Self-tuning control of an ore crusher'. Proceedings of the IFAC stochastic control symposium, Budapest, 1974, pp. 491–497
6 HETTHESSY, J., and KEVICZKY, L.: 'Some innovations to the minimum variance control'. Proceedings of the IFAC stochastic control symposium, Budapest, 1974, pp. 353–361
7 ÅSTRÖM, K.J., and WITTENMARK, B.: 'Analysis of a self-tuning regulator for nonminimum phase systems'. Proceedings of the IFAC stochastic control symposium, Budapest, 1974, pp. 165–173
8 ASHTON, R.P.: 'Simulation study of a k-step-ahead control law'. Oxford University Internal Report, 1973
9 CLARKE, D.W., and HASTINGS-JAMES, R.: 'Design of digital controllers for randomly disturbed systems', *Proc. IEE*, 1971, **118**, (10), pp. 1503–1506
10 CLARKE, D.W., DYER, D.A.J., HASTINGS-JAMES, R., ASHTON, R.P., and EMERY, J.B. 'Identification and control of a pilot scale boiling rig'. proceedings of 3rd IFAC symposium on system identification, The Hague/Delft, 1, pp. 355–366
11 ÅSTRÖM, K.J.: 'Introduction to stochastic control theory' (Academic Press, 1970)
12 DURBIN, J.: 'Estimates of parameters in time series regression models', *J. R. Stat. Soc. B*, 1960, **22**, pp. 139–153

## 8 Appendix

### *k*-step-ahead predictor

The system, eqn. 1 and Fig. 1, may be written in $z$-transform form as

$$y_{t+k} = \frac{B(z^{-1})}{A(z^{-1})} u_t + z^k \frac{C(z^{-1})}{A(z^{-1})} \xi_t + \frac{1}{A(z^{-1})} d \qquad (17)$$

The disturbance term may be considered as the sum of two terms, one representing future disturbances, and the other disturbances that have occured up to and including time $t$. Explicitly, it may be written as:

$$z^k \frac{C}{A} \xi_t = z^k E(z^{-1}) \xi_t + \frac{F(z^{-1})}{A} \xi_t = e_{t+k} + e'_t \qquad (18)$$

where $E(z^{-1}) = \sum_0^{k-1} e_i z^{-i}$

and $F(z^{-1}) = \sum_0^{n-1} f_i z^{-i}$

Combining eqns. 17 and 18

$$y_{t+k} = e_{t+k} + \left[\frac{B}{A} u_t + \frac{F}{A} \xi_t + \frac{1}{A} d\right] \qquad (19)$$

By comparison of eqns. 19 and 20 it follows that:

$$y^*_{t+k|t} = \frac{B}{A} u_t + \frac{F}{A} \xi_t + \frac{1}{A} d \qquad (20)$$

Now from eqns. 18 and 14 we have:

$$\xi_t = \frac{e_t}{A} = \frac{y_t - y^*_{t|t-k}}{A}$$

so eqn. 20 becomes

$$y_{t+k|t} = \frac{B}{A} u_t + \frac{F}{EA}(y_t - y^*_{t|t-k}) + \frac{d}{A}$$

Hence, using the identity eqn. 18

$$\frac{C}{EA} y^*_{t+k|t} = \frac{B}{A} u_t + \frac{F}{EA} y_t + \frac{d}{A}$$

i.e.

$$C y^*_{t+k|t} = EBu_t + Fy_t + Ed$$

As $d$ is a constant

$$E(z^{-1})d = E(1)d = \gamma d$$