Closed Form Solution for Optimal Buffer Sizing Using The Weierstrass Elliptic Function

Abstract

This paper presents a fundamental result on buffer sizing. Given an interconnection wire with n buffers evenly spaced along the wire, we would like to size all buffers such that the Elmore delay is minimized. It is well known that the problem can be solved by an iterative algorithm which sizes one buffer at a time. However, no closed form solution has ever been reported. In this paper, we present a closed form solution to the problem. We derive a buffer sizing function f(x) where f(x) gives the optimal buffer size for the buffer at position x. We show that f(x) can be expressed in terms of the Weierstrass elliptic function $\wp(x)$ *and its derivative* $\wp'(x)$.

1. Introduction

Buffer insertion, buffer sizing and wire sizing have been shown to be effective techniques in reducing interconnect delay [1]. This paper focuses on the buffer sizing problem under the Elmore delay model [2]. Suppose we only have one buffer size and we want to insert a fixed number of buffers into an interconnection wire for delay minimization, it is well known that the buffers will be uniformly spaced in the optimal solution [3, 4]. Clearly if we allow to size the buffers, the overall delay will be further reduced. Therefore, this paper considers the following buffer sizing problem:

Given an interconnection wire with n buffers evenly spaced along the wire, we would like to size all buffers such that the Elmore delay is minimized. It is well known that the problem can be solved by an iterative algorithm which sizes one buffer at a time [5, 6]. However, no closed form solution has ever been reported.

In this paper, we present a closed form solution to the buffer sizing problem. Without loss of generality, we may assume that the interconnection wire is of unit length represented by the interval $[0, 1]$ with source (driver) at 0 and sink (load) at 1. Let $x_i = \frac{i}{n+1}$ be the position of buffer i, for $1 \leq i \leq n$. We derive a continuous function $f(x)$ such that $f(x_i)$ gives the optimal buffer size for buffer i, for all i. The buffer sizing function $f(x)$ can be expressed in terms of the Weierstrass elliptic function $\wp(x)$ and its derivative $\wp'(x)$ as follows:

$$f(x) = a + \frac{b\wp'(x) + c\wp(x) + d}{2(\wp(x) - e)^2}, \qquad (1)$$

where a, b, c, d, and e are constants.

The reminder of the paper is organized as follows. Section 2 presents our circuit model and derives a recurrence relation for optimal buffer sizing. Section 3 shows that the recurrence relation for optimal buffer sizing implies an ordinary differential equation. In Section 4, we give a brief overview of the Weierstrass $\wp(x)$ function and its fundamental difference equation that will be used in Section 5 to give our closed form expression (1). Section 6 briefly discusses an integration constant that arose during our derivations and in Section 7, we show our experimental results. Finally we conclude the paper in Section 8.

2. The recurrence relation

This section introduces our circuit model of an interconnect with several equally spaced buffers. We derive a recurrence relation for the buffer sizes and show that it is both a necessary and sufficient condition to minimize the Elmore delay on the wire.

Consider an interconnect of length L that has a total resistance of R and capacitance of C. The interconnect has a driver resistance R_D at the source and a load capacitance C_L at the sink. To minimize the propagation delay on the interconnect, we want to insert n buffers at equally spaced locations, thus we split the interconnect into $n+1$ segments; each segment has a length of $\frac{L}{n+1}$.

We have $R_S = \frac{R}{n+1}$ and $C_S = \frac{C}{n+1}$ being the segment resistance and capacitance, respectively. (Since we do not perform wire sizing, we can include the fringing capacitance into the unit capacitance of our wire model and can thus avoid limitations that occured, for example, in [7]). Each wire segment is modeled as a π-type RC-circuit.

A buffer of size b is represented by a switch-level RC-circuit as shown in Figure 1, where R_B/b denotes its output resistance, $C_B b$ denotes its input capacitance and $C_D b$ denotes its output capacitance. For the ease of presentation, we also model the driver and the load as buffers of

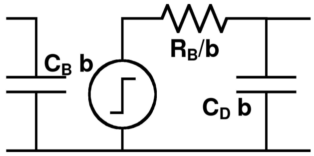

Figure 1: RC switch-level model of a buffer of size b

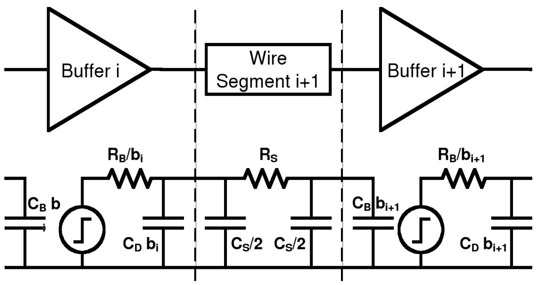

Figure 2: RC switch level interconnect between buffers i and $i+1$

fixed sizes $b_0 = \lambda$ and $b_{n+1} = \mu$, respectively, so that $R_D = R_B/b_0$ and $C_L = C_B b_{n+1}$.

The Elmore delay between buffer i and buffer $i+1$ is, as shown in Figure 2,

$$ED_i = \frac{R_B}{b_i}(C_D b_i + C_S + C_B b_{i+1}) + R_S(\frac{C_S}{2} + C_B b_{i+1}) \tag{2}$$

and the total delay from the driver to the load is

$$ED = \sum_{i=0}^{n} ED_i. \tag{3}$$

Clearly, (3) is only a function of $b_1,\ b_2,\ ...,\ b_n$ since all other parameters depend on the physical device characteristics. As it was stated above, $b_0 = \lambda$ and $b_{n+1} = \mu$ define boundary conditions because the driver and the load are fixed. The problem therefore consists of finding positive $b_1^*,\ b_2^*,\ ...,\ b_n^*$ that minimize (3).

A neccessary condition is that the partial derivatives of ED with respect to all b_i are equal to zero, i. e.,

$$\frac{\partial}{\partial b_i} ED = 0 \quad \forall i = 1...n. \tag{4}$$

Since b_i only appears in the expressions of ED_{i-1} and ED_i, we get

$$R_S C_B + R_B C_B \frac{1}{b_{i-1}} - R_B C_S \frac{1}{b_i^2} - R_B C_B \frac{b_{i+1}}{b_i^2} = 0. \tag{5}$$

Solving for b_i yields our basic recurrence relation for optimal buffer sizing,

$$b_i^2 \quad = \quad \frac{b_{i+1} + C_S/C_B}{b_{i-1}^{-1} + R_S/R_B} \quad \forall i = 1, ..., n. \tag{6}$$

To simplify the notation in the following sections, we define

$$\frac{\alpha}{n+1} = \frac{C_S}{C_B} = \frac{C}{(n+1)C_B} \tag{7}$$

and

$$\frac{\beta}{n+1} = \frac{R_S}{R_B} = \frac{R}{(n+1)R_B}. \tag{8}$$

Then (6) becomes

$$b_i^2 = \frac{b_{i+1} + \alpha/(n+1)}{b_{i-1}^{-1} + \beta/(n+1)}. \tag{9}$$

Note that (9) gives a recurrence relation for the buffer sizes with b_i expressed as a function of b_{i-1} and $b_{i+1}$with boundary values $b_0 = \lambda$ and $b_{n+1} = \mu$. Furthermore, it is both a necessary and sufficient condition to minimize the Elmore delay expression (3). To see this, we note that (3) is of the form

$$ED(b_1,\ ...,\ b_n) = \sum_{i=1}^{T} a_i \prod_{j=1}^{n} b_j^{c_{ij}} \tag{10}$$

where the a_i are non-negative, b_j are positive and c_{ij} are real numbers. Equation (10) is a posynomial with T terms and n variables. Under a change of variables $b_i = e^{d_i}$, $ED(d_1,\ ...,\ d_n)$ is a convex function of $(d_1,\ ...,\ d_n)$. Therefore, the local optimum determined by (9) is simultanously a global optimum.

3. ODE Formulation

In the following sections, we derive our closed form expression $f(x)$. First, we show that the recurrence relation for optimal buffer sizing (9) implies a second order ordinary differential equation. We will use this ODE then in Section 5 to find the buffer sizing function $f(x)$.

In a first step, we replace b_i in (9) by $f(x_i)$ and have

$$f(x)^2 = \frac{f(x_{i+1}) + \alpha/(n+1)}{f(x_{i-1})^{-1} + \beta/(n+1)} \tag{11}$$

Next we set $x_i = x$ and $\Delta x = \frac{1}{n+1}$. We have $x_{i+1} = x + \Delta x$, $x_{i-1} = x - \Delta x$, and $\frac{1}{n+1} \cdot \frac{1}{n} \approx \Delta x^2$. Equation (11) becomes

$$f(x)^2 = \frac{f(x+\Delta x) + \Delta x^2 n\alpha}{f(x-\Delta x)^{-1} + \Delta x^2 n\beta}. \tag{12}$$

To show that the RHS of (12) effectively defines a second order ordinary differential equation, it is useful to substitute $f(x) = e^{g(x)}$ and to perform a Taylor expansion of the resulting exponential terms on the RHS, so we have

$$e^{g(x+\Delta x)} = \\ e^{g(x)} + \Delta x g' e^g + \frac{1}{2}\Delta x^2 e^g (g'' + g'^2) + O(\Delta x^3) \quad (13)$$

for the numerator and a similar expression for the denominator. Collecting terms, (12) becomes the quotient of two polynomials in Δx,

$$f(x)^2 = e^{2g(x)} = \\ \frac{e^g(1 + \Delta x g' + \Delta x^2(\frac{1}{2}g'^2 + \frac{1}{2}g'' + \alpha n e^{-g})) + O(\Delta x^3)}{e^{-g}(1 + \Delta x g' + \Delta x^2(\frac{1}{2}g'^2 - \frac{1}{2}g'' + \beta n e^g)) + O(\Delta x^3)} \\ = \frac{P(\Delta x)}{Q(\Delta x)} = A(\Delta x). \quad (14)$$

It is now our goal to find $A(\Delta x) = a_0 + a_1\Delta x + a_2\Delta x^2 + O(\Delta x^3)$ so that $f(x)^2 = A(\Delta x)$. The a_i are determined by

$$(a_0 + a_1\Delta x + a_2\Delta x^2 + ...)(q_0 + q_1\Delta x + q_2\Delta x^2 + ...) \\ = p_0 + p_1\Delta x + p_2\Delta x^2 + O(\Delta x^3). \quad (15)$$

and we simply have to compare coefficients to get

$$\begin{aligned} a_0 &= \frac{p_0}{q_0} = e^{2g} && (16) \\ a_1 &= \frac{1}{q_0}(p_1 - a_0 q_1) = e^{2g}g' - e^{2g}g' = 0 && (17) \\ a_2 &= \frac{1}{q_0}(p_2 - a_0 q_2 - a_1 q_1) \\ &= g'' e^{2g} + \alpha n e^g - \beta n e^{3g} \\ &= e^{2g}(g'' + \alpha n e^{-g} - \beta n e^g). && (18) \end{aligned}$$

Hence, (14) becomes

$$A(\Delta x) = e^{2g}(1 + \Delta x^2(g'' + \alpha n e^{-g} - \beta n e^g)) + O(\Delta x^3) \\ = e^{2g(x)} = f(x)^2 \quad (19)$$

For the last step, we drop the terms of third and higher order and require that $g(x)$ satisfies the differential equation $g'' = \beta n e^g - \alpha n e^{-g}$.

An intermediate result is that the buffer sizing function $f(x)$ must satisfy

$$f(x) = e^{g(x)}, \quad x = \frac{i}{n+1} \quad \forall i = 0, ..., n+1 \quad (20)$$

where $g(x)$ is a solution to the second-order ordinary differential equation

$$g'' = \beta n e^g - \alpha n e^{-g} \quad (21)$$

with the boundary conditions

$$g(0) = \ln(\lambda) \text{ and } g(1) = \ln(\mu). \quad (22)$$

4. Weierstrass elliptic function

Our buffer sizing function $f(x)$ will be obtained by solving an elliptic integral and has the form of a rational function in terms of the Weierstrass $\wp$-function and its derivative $\wp'$. Before we solve the ODE from the last section, we briefly state important properties of the Weierstrass elliptic function.

Apart from being a prototype for all elliptic functions, the Weierstrass $\wp$-function has the fundamental property that it satisfies the differential equation

$$(\wp'(x))^2 = 4\wp^3(x) - g_2\wp(x) - g_3. \quad (23)$$

$\wp(x)$ and its derivative $\wp'(x)$ have two parameters, g_2 and g_3, which are called *invariants*. We will use the shorter notation $\wp(x)$ and $\wp'(x)$ instead of $\wp(x;\, g_2,\, g_3)$ and $\wp'(x;\, g_2,\, g_3)$ in this paper.

The elliptic integral

$$x = \int_y^\infty (4t^3 - g_2 t - g_3)^{-\frac{1}{2}}\, dt \quad (24)$$

implicitely defines y as a function of x and cannot be solved using elementary functions. However, differentiating leads to

$$(\frac{dy}{dx})^2 = 4y^3 - g_2 y - g_3 \quad (25)$$

which is of the same form as (23). The solution is then given by $y = \wp(x)$ with g_2, g_3 as parameters [8]. This relationship can be used to solve elliptic integrals of the form

$$x = \int_{y_0}^{y} \frac{dt}{\sqrt{h(t)}}, \quad (26)$$

where $h(t)$ is a cubic or quartic polynomial. Provided that y_0 is a root of $h(t)$, the solution to (26) is given by [8]

$$y = y_0 + \frac{\frac{1}{4}h'(y_0)}{\wp(x) - \frac{1}{24}h''(y_0)}, \quad (27)$$

where $\wp(x)$ is dependant on the invariants g_2 and g_3 of the polynomial $h(t)$. However, for our purposes, the more general formula given by Weierstrass [8] allows the lower bound y_0 of the integral (26) to be any constant a, not necessarily a root of $h(t)$. The solution is then slightly more complex, but it is still only a rational function involving $\wp(x)$, $\wp'(x)$ and some constants,

$$y = a + \frac{\sqrt{h(a)}\wp'(x)}{2(\wp(x) - \frac{1}{24}h''(a))^2 - \frac{1}{48}h(a)h^{iv}(a)} \\ + \frac{\frac{1}{2}h'(a)(\wp(x) - \frac{1}{24}h''(a)) + \frac{1}{24}h(a)h'''(a)}{2(\wp(x) - \frac{1}{24}h''(a))^2 - \frac{1}{48}h(a)h^{iv}(a)}. \quad (28)$$

Given the invariants g_2 and g_3 of $h(t)$ as parameters, $\wp$ and $\wp'$ can be calculated efficiently and are also available in standard mathematical software such as Mathematica. The derivations of (27) and (28) are rather lengthy and therefore omitted from this paper.

In the rest of this paper, we derive an elliptic integral of the form (26) from the set of recurrence relations (9). Its explicit solution $f(x)$ is our optimal buffer sizing expression and has the form of (28).

5. Closed form solution

In this section, we give the closed form expression $f(x)$ for optimal buffer sizing which has the form of a rational function in terms of the Weierstrass functions $\wp(x)$ and $\wp'(x)$. We have found so far that $f(x) = e^{g(x)}$, where $g(x)$ satisfies the ODE (21). We will now derive an ODE in f which finally can be solved using the Weierstrass $\wp$-function.

As a first step, we multiply both sides of (21) by $2g'$ and apply the chain rule of differentiation backwards. Hence,

$$\begin{aligned} 2g'g'' &= 2g'(n\beta e^g - n\alpha e^{-g}) && (29)\\ \frac{d}{dx}(g'^2) &= 2n\frac{d}{dx}(\beta e^g - \alpha e^{-g}) && (30)\\ g'^2 &= 2n(\beta e^g - \alpha e^{-g}) + c, && (31) \end{aligned}$$

where c is a constant of integration that has to be determined numerically. We briefly deal with this problem in Section 6.

We now undo the substitution made in (14) so that $g(x) = \ln f(x)$ and therefore $g' = f'(x) \cdot \frac{1}{f(x)}$. The desired buffer sizing function $f(x)$ is now given as the solution to the differential equation

$$f'^2 = \left(\frac{df}{dx}\right)^2 = 2n\beta f^3 + cf^2 + 2n\alpha f. \qquad (32)$$

Moving all terms with f to one side and integrating gives an elliptic integral

$$\int dx = x = \int_{f(0)}^{f(x)} \frac{dt}{\pm\sqrt{2n\beta t^3 + ct^2 + 2n\alpha t}} \qquad (33)$$

which essentially has the same form as (26). It only remains to find the invariants g_2 and g_3 in order to apply (28) to get a solution for $f(x)$. This can be done by a simple transform of variables

$$t' = \frac{2}{n\beta} \cdot t - \frac{c}{6n\beta} \qquad (34)$$

in (33). This changes the expression under the radical sign

$$2\beta n t^3 + ct^2 + 2\alpha n t \qquad (35)$$

into

$$4t'^3 - (\frac{c^2}{12} - \alpha\beta n^2)t' - (\frac{1}{12}\alpha\beta n^2 c - \frac{c^3}{216})$$

so that the invariants become

$$g_2 = \frac{c^2}{12} - \alpha\beta n^2, \quad g_3 = \frac{1}{12}\alpha\beta n^2 c - \frac{c^3}{216}. \qquad (36)$$

This leads to our central result. The buffer sizing function $f(x)$ for optimal buffer sizing at equally spaced points $x = x_i = \frac{i}{n+1},\ i = 0...n+1$, is given by

$$\begin{aligned} f(x) = \lambda &+ \frac{\sqrt{h(\lambda)}\wp'(x)}{2(\wp(x) - \frac{1}{24}h''(\lambda))^2} \\ &\frac{+\frac{1}{2}h'(\lambda)(\wp(x) - \frac{1}{24}h''(\lambda)) + \frac{1}{24}h(\lambda)h'''(\lambda)}{2(\wp(x) - \frac{1}{24}h''(\lambda))^2}, \qquad (37) \end{aligned}$$

where $h(t) = 2\beta n t^3 + ct^2 + 2\alpha n t$ and the Weierstrass function has the invariants (36) as parameters. $f(x)$ fulfills the boundary values $f(0) = \lambda$ and $f(1) = \mu$.

6. Integration constant

In the last section, we did not give a description of the integration constant c. For completeness, this section shows that one can distinguish four non-trivial cases for the behaviour of $f(x)$. Each case has its different expression that determines c.

1. From (21), g'' can be either purely non-negative, then $g(x)$ and consequently $f(x) = e^{g(x)}$ are convex functions and $f(x)$ has the shape of a U, limited by the boundary values. Setting $g'' = 0$, we find that all b_i, including $b_0 = \lambda$ and $b_{n+1} = \lambda$, must be larger than $e^{\ln\sqrt{\frac{\alpha}{\beta}}} = \sqrt{\frac{\alpha}{\beta}}$. The slope f' from (32) must consist of a decreasing part from λ to a minimum buffer and an increasing part from the minimum to μ. It is not hard to see that the value of this minimum is given by the zero of (32) that lies to the right of $\sqrt{\frac{\alpha}{\beta}}$ which is

$$b_{min}(c) = \frac{-c + \sqrt{c^2 - 16\alpha\beta n^2}}{4\beta n} \qquad (38)$$

   We then have a simple condition that the constant c must fulfill: integrating and summing the two parts of the slope must be equal to one,

$$\begin{aligned} 1 \;=\; & \int_{b_{min}(c)}^{\lambda} \frac{dy}{\sqrt{2n\beta y^3 + cy^2 + 2n\alpha y}} \\ + \; & \int_{b_{min}(c)}^{\mu} \frac{dy}{\sqrt{2n\beta y^3 + cy^2 + 2n\alpha y}}. \qquad (39) \end{aligned}$$

2. Conversely, if g'' is strictly non-positive, $f(x)$ is concave and all b_i are below $\sqrt{\frac{\alpha}{\beta}}$. This case is analog to the first one and leads to a similar condition.

3. Also, f' can have only one monotonic part and is strictly increasing on the whole interval $[\lambda;\mu]$, then $\lambda \leq \sqrt{\frac{\alpha}{\beta}}$ and $\mu \geq \sqrt{\frac{\alpha}{\beta}}$. We integrate the (positive) slope f' from λ to μ and have a simpler condition for c,
$$1 = \int_{\lambda}^{\mu} \frac{dy}{\sqrt{2n\beta y^3 + cy^2 + 2n\alpha y}}. \tag{40}$$

4. The case that f' is strictly decreasing and thus $\lambda \geq \sqrt{\frac{\alpha}{\beta}}$ and $\mu \leq \sqrt{\frac{\alpha}{\beta}}$ can be handled in a similar manner as the previous one.

5. Note that if f' is equal to zero, $f(x) = \sqrt{\frac{\alpha}{\beta}}$ follows as a trivial case.

For the cases 1-4, one can construct simple procedures to find c. For case 3, for example, one can interpret (40) as a function

$$d(c) = \int_{\lambda}^{\mu} \frac{dy}{\sqrt{2n\beta y^3 + cy^2 + 2n\alpha y}} - 1. \tag{41}$$

It is not hard to show that $d(c)$ is monotonic with respect to c and has a unique root which can be determined efficiently by bisection, for example.

7. Experimental results

In this section, we show that our solution performs very well in practice. We compare buffer sizes obtained by an iterative method [5, 6] with values calculated by evaluating (37) at equally spaced points on the interval $[0, 1]$.

For the physical parameters, we rely on values given in [1]. More specifically, we consider a copper wire ($\rho = 2.2\mu\Omega \cdot cm$) with a width of 130 nm, a length of 1 mm and an aspect ratio of 2.1:1. We add unit-length area and fringing capacitance together. These values are summarized in table 1.

Table 1: Physical parameters for unit buffer and wire

Buffer		Wire	
Output res. [kΩ]	**4.5**	**resistance [$\frac{\Omega}{mm}$]**	**620**
Input cap. [fF]	**0.425**	**capacitance [$\frac{fF}{mm}$]**	**58.5**
intrinsic delay [ps]	**45.8**	**length [mm]**	**1**

Applying (7) and (8), we have $\alpha = 137.65$ and $\beta = 0.1378$. The following Figures (3) - (6) plot iteratively obtained buffer sizes (dots) and those calculatd by applying (37) (straight curve) for several combinations of b_0, b_{n+1} and n.

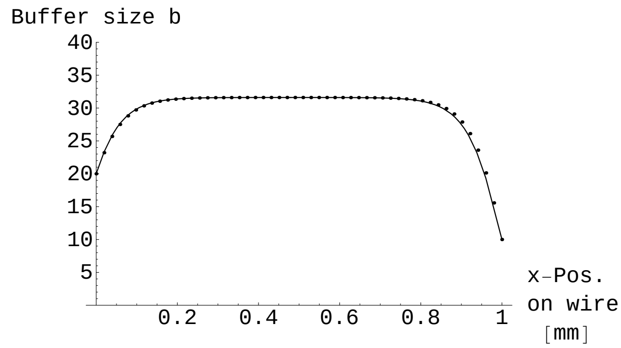

Figure 3: $n = 50$, $b_0 = 20$, $b_{51} = 10$

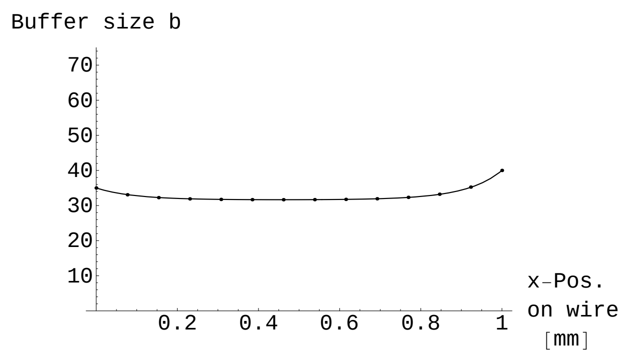

Figure 4: $n = 12$, $b_0 = 35$, $b_{13} = 40$

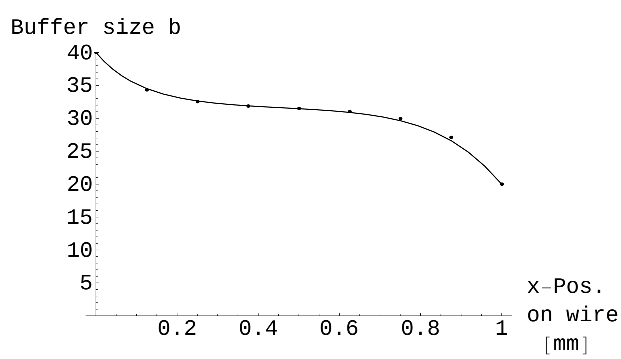

Figure 5: $n = 7$, $b_0 = 40$, $b_8 = 20$

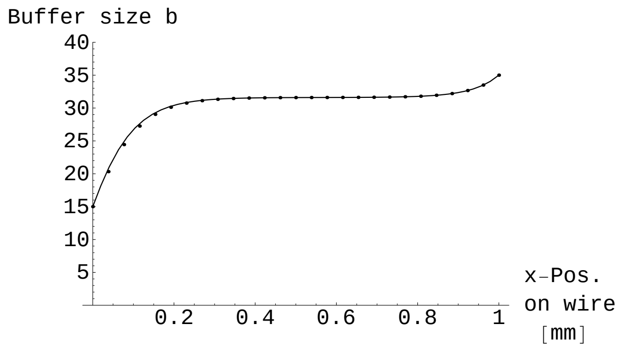

Figure 6: $n = 25$, $b_0 = 15$, $b_{26} = 35$

We calculated many different cases and observed that our closed form expression shows a very good agreement with iteratively calculated buffer sizes.

8. Conclusion

In this paper we addressed the problem of sizing n uniformly spaced buffers on an interconnection wire to minimize Elmore delay.

Previously there was no known closed form solution to this problem. We presented a closed-form buffer sizing function $f(x)$, expressed in terms of the Weierstrass elliptic function $\wp(x)$ and its derivative $\wp'(x)$, such that $f(x_i)$ gives the optimal buffer size for buffer i (at position x_i), $1 \leq i \leq n$.

We showed that the buffer sizes obtained by our closed form expression matched very well with those obtained by the iterative method [5, 6].

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