Is Normal Distribution Accurate Enough for Parametric Variation Modeling?

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1. MOTIVATION

In statistical static timing analysis (SSTA), the maximum function is a pain in terms of computing the probability density function (PDF).

In this section, we derive closed formula to compute the PDF of a maximum of two normal random variables with correlation. We show that the newly computed PDF falls into a more general class of normal distribution: called "skewed-normal" distribution.

We argue that skewed-normal distribution should be used for more accurate parametric variation modeling. New SSTA framework should employ skewed-normal distribution, instead of purely normal distribution.

Assume (X, Y) are two random variables with correlation that can be modeled as a bivariate normal distribution, i.e.,

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \cdot exp(-\frac{x^2 - 2\rho \cdot x \cdot y + y^2}{2(1-\rho^2)}) \tag{1}$$

with the cumulative density function (CDF) given as

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y) \cdot dx \cdot dy. \tag{2}$$

Our question would be: if $Z = max(X, Y)$, what will be the PDF for Z?

To derive the PDF of Z, we compute the CDF of Z first. Because $G(z) = P(Z \leq z) = P(X \leq z, Y \leq z) = F(z, z)$. Then we can compute the PDF of Z by taking the derivative of $G(z)$, i.e.,

$$\begin{aligned} g(z) &= \frac{dG}{dz} \\ &= \frac{dF(z,z)}{dz} \\ &= \frac{d}{dz}\int_{-\infty}^{z}\int_{-\infty}^{z} f(x,y) \cdot dx \cdot dy \\ &= \int_{-\infty}^{z} f(z,y) \cdot dx + \int_{-\infty}^{z} f(x,z) \cdot dy \\ &= 2 \cdot \int_{-\infty}^{z} \frac{1}{2\pi(1-\rho^2)^{1/2}} exp(-\frac{z^2 - 2\rho \cdot z \cdot y + y^2}{2(1-\rho^2)}) \cdot dy \\ &= 2 \cdot \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2}) \cdot \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi \cdot (1-\rho^2)}} \cdot exp(-\frac{(y-\rho z)^2}{2 \cdot (1-\rho^2)}) dy \\ &= 2 \cdot \phi(z) \cdot \Phi(z\sqrt{\frac{1-\rho}{1+\rho}}) \end{aligned} \tag{3}$$

where $\phi(z)$ is the PDF of a normal distribution, while $\Phi(z\sqrt{\frac{1-\rho}{1+\rho}})$ is the CDF of a normal distribution taking value at $z\sqrt{\frac{1-\rho}{1+\rho}}$.

Equation says that the PDF of Z, which is the maximum of X and Y, equals to two times of the product of a normal PDF and a CDF at some particular values. This kind of PDF indeed falls into another more general normal distrituion, called "skewed-normal" distribution, which emerges in recent statistics researches.

2. REFERENCES