Is Normal Distribution Accurate Enough for Parametric Variation Modeling?

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1. MOTIVATION

In statistical static timing analysis (SSTA), the maximum function is a pain in terms of computing the probability density function (PDF).

In this section, we derive closed formula to compute the PDF of a maximum of two normal random variables with correlation. We show that the newly computed PDF falls into a more general class of normal distribution: called "skewed-normal" distribution.

We argue that skewed-normal distribution should be used for more accurate parametric variation modeling. New SSTA framework should employ skewed-normal distribution, instead of purely normal distribution.

Assume (X, Y) are two random variables with correlation that can be modeled as a bivariate normal distribution, i.e.,

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \cdot exp(-\frac{x^2 - 2\rho \cdot x \cdot y + y^2}{2(1-\rho^2)}) \quad (1)$$

with the cumulative density function (CDF) given as

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y) \cdot dx \cdot dy. \quad (2)$$

Our question would be: if $Z = max(X, Y)$, what will be the PDF for Z?

To derive the PDF of Z, we compute the CDF of Z first. Because $G(z) = P(Z \le z) = P(X \le z, Y \le z) = F(z, z)$. Then we can compute the PDF of Z by taking the derivative of $G(z)$, i.e.,

$$\begin{aligned}
g(z) &= \frac{dG}{dz} \\
&= \frac{dF(z,z)}{dz} \\
&= \frac{d}{dz}\int_{-\infty}^{z}\int_{-\infty}^{z} f(x,y) \cdot dx \cdot dy \\
&= \int_{-\infty}^{z} f(z,y) \cdot dx + \int_{-\infty}^{z} f(x,z) \cdot dy \\
&= 2 \cdot \int_{-\infty}^{z} \frac{1}{2\pi(1-\rho^2)^{1/2}} exp(-\frac{z^2 - 2\rho \cdot z \cdot y + y^2}{2(1-\rho^2)}) \cdot dy \\
&= 2 \cdot \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2}) \cdot \\
&\quad \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi \cdot (1-\rho^2)}} \cdot exp(-\frac{(y-\rho z)^2}{2 \cdot (1-\rho^2)}) dy \\
&= 2 \cdot \phi(z) \cdot \Phi(z\sqrt{\frac{1-\rho}{1+\rho}}) \quad (3)
\end{aligned}$$

where $\phi(z)$ is the PDF of a normal distribution, while $\Phi(z\sqrt{\frac{1-\rho}{1+\rho}})$ is the CDF of a normal distribution taking value at $z\sqrt{\frac{1-\rho}{1+\rho}}$.

Equation says that the PDF of Z, which is the maximum of X and Y, equals to two times of the product of a normal PDF and a CDF at some particular values. This kind of PDF indeed falls into another more general normal distribution, called "skewed-normal" distribution, which emerges in recent statistics researches.

Now we give the formal definition of skewed-normal distribution as follows:

DEFINITION 1. **Skewed-normal distribution** *is a distribution of random variable Z with the PDF given as*

$$g(z) = 2 \cdot \phi(z) \cdot \Phi(\alpha \cdot z) \quad (4)$$

where $\phi(z)$ and $\Phi(z)$ are PDF and CDF of a standard normal distribution $N(0, 1)$, and α is a skew factor (shape parameter).

To appreciate why α is called a skew factor for the skewed-normal distribution, we plot three different $g(z)$ with different skew factors in Figure 1. From the figure, we can see that when $\alpha = 0$, $g(z)$ becomes the standard normal distribution $N(0, 1)$; when $\alpha < 0$, $g(z)$ is negatively skewed; and when $\alpha > 0$, $g(z)$ is positively skewed. In another word, skewed-normal distribution is a more general class of PDF that includes the standard normal distribution as a special

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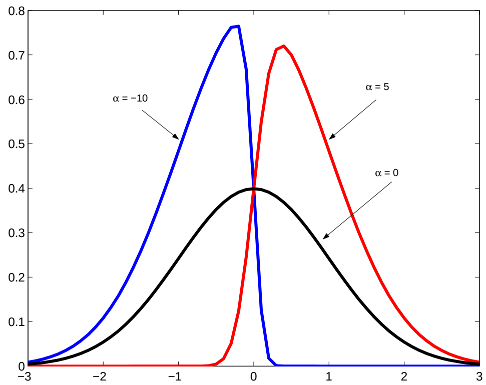

Figure 1: Illustration of skewed-normal distribution.

case because of the extra freedom given by the skew factor α.

Because the PDF of the maximum of two normal variables (1) has the exact form as (4) with $\alpha = \sqrt{\frac{1-\rho}{1+\rho}}$, we would expect that the PDF should also exhibit the skewness as shown in Figure 1. However, current SSTA practice is to model the maximum function as a normal distribution owning in part to the factor that they may not aware of the closed form solution as shown in (1) and the difficulty in obtaining a general formula for the maximum function of two arbitrary distribution.

Based upon the above observation, we believe that skewed-normal distribution should be used as a more general PDF model for parametric variations in order to obtain accurate SSTA. The reasons are multi-fold: (1) skewed-normal distribution is the exact formula when we take the maximum of two normal variations; (2) as we shall see in the next section, skewed-normal distribution is also suitable for device parametric characterization; (3) skewed-normal distribution have many properties that are in parallel with the conventional normal distribution, which makes the mathematical manipulation of these random variables as easy as the conventional normal distribution but with large flexibility; and (4) we believe that for most systematic variations, skewness is more prevalent than symmetry in general. For example, a large portion of process variation is due to the difficulty in print the small features exactly. However, the difficulty should become less prominent as the geometry size becomes increasingly large.

2. RANDOM L_{EFF} INDUCED DELAY VARIATION

In this section, we extract the device delay parametric variation in the presence of process variation via Monte Carlo simulation. Because of the lack of access to the real sources of foundry process variations, we only model the random L_{eff} variation using $65nm$ BSIM model in this section. Moreover, we assume the variation of L_{eff} to be a symmetric normal distribution to represent the common wisdom regarding to process variations. However, we will show that even under this very simple and symmetric setting, the skewness of delay variation is omnipresent.

A minimum size two-stage buffer is used for Hspice simulation in order to extract the gate delay parametric. The

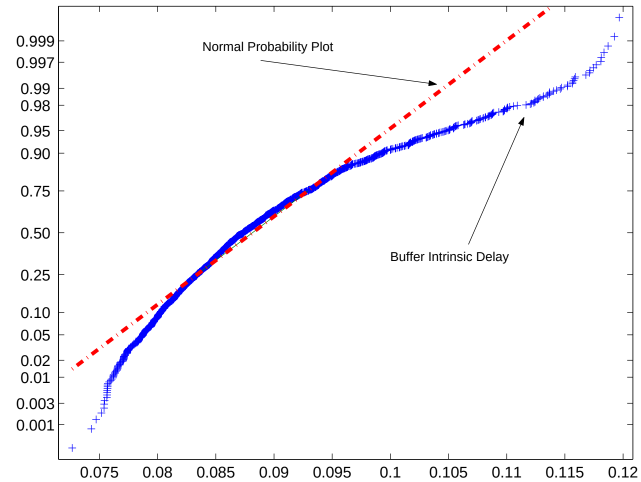

Figure 2: Normal probability plot of the delay distribution.

standard deviation of L_{eff} variation is set as 10% of the mean value in our experiment. We run Monte Carlo simulation 2000 times and measure the intrinsic delay of buffers when different L_{eff} values are assumed.

To test whether or not the delay parametric variation is a normal distribution, we show the normal probability plot in Figure 2, where each cross dot is a measured delay from Hspice simulation. The dashed line represents a normal distribution by assuming the underlying PDF of delay is a normal distribution. Any data that deviate from the straight line defy the normality assumption. It is apparent from the figure that the delay is not a normal distribution.

We further plot the PDF of the collected delay data in Figure 3. Apparently, the PDF shows skewness with a long tail to the right. To model such a distribution in a closed mathematical form, people always assume a known type of PDF for the data, and then do a maximum-likelihood curve fitting to decide the exact formula for the PDF. Normal distribution has been assumed extensively to model parametric variations in literature due to its simplicity and well-known properties. However, as we have already seen in Figure 3 that the skewness of distribution is not negligible and should be captured correctly in order for accurate timing diagnostics and yield prediction.

To illustrate this point, we employ the maximum-likelihood curve fitting technique to determine the PDF of delay variation. Normal distribution and skewed-normal distribution are assumed respectively. After obtaining the two fitted PDFs, we overlay them in the same plot as shown in Figure 3. According to Figure 3, we can see that the skewed-normal PDF can capture both the overall distribution and the long tailed distribution of simulated data very well. On the contrary, the normal PDF can barely capture the overall distribution, not to say the long tail skewness.

3. REFERENCES

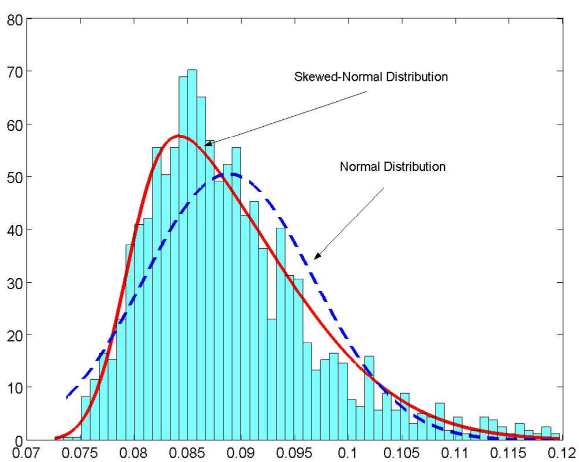

Figure 3: PDF for the simulated data, with a normal distribution fitting and a skewed-normal distribution fitting.