Model Reduction of Systems with Symmetries

Bart Vanluyten, Jan C. Willems and Bart De Moor

***Abstract*— In this paper we address the problem of approximating symmetric systems with systems with the same symmetry. We show that for periodic systems, a reduced order periodic system can be obtained by SVD-techniques. We also show that pointwise symmetries of the impulse response are retained after balanced model reduction. Both results are based on the fact that under certain conditions the SVD-reduction of a matrix with unitary symmetries leads to a lower rank matrix with the same symmetries. The results are applied to model reduction of an interconnected system.**

I. INTRODUCTION

Model reduction is undoubtedly one of the most useful aspects of system theory because of its immediate relevance to model simplification. It combines mathematical modeling problems with computational complexity issues, two of the pillars of modern applied mathematics. However, physical models usually have some properties which are very important from the physical point of view, as conservativeness, dissipativity, etc. Also symmetries fall into this category. This is the topic of the research domain in which this article falls: How can we reduce a symmetric model and obtain a reduced model that preserves the symmetry?

II. SYSTEMS WITH SYMMETRIES

We consider linear time-invariant input-output systems in discrete time, described by

$$\begin{array}{rcl} x(t+1) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t), \end{array} \qquad (\mathscr{S})$$

with $u(t) \in \mathbb{R}^{\mathtt{m}}$, $y(t) \in \mathbb{R}^{\mathtt{p}}$, and $x(t) \in \mathbb{R}^{\mathtt{n}}$, or equivalently

$$y(t) = \sum_{\tau=1}^{\infty} H(\tau)u(t-\tau), \qquad (\mathscr{S})$$

with $H(t) = CA^{t-1}B$, $t \in \mathbb{N}$ the Markov parameters of the system. Associated with this system is the (doubly infinite) block Hankel matrix

$$\mathfrak{H}_H = \begin{bmatrix} H(1) & H(2) & H(3) & \cdots \\ H(2) & H(3) & H(4) & \cdots \\ H(3) & H(4) & H(5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We will consider dynamic symmetries from a rather concrete point of view (an abstract theory may be found in [2]). We start by giving some examples of symmetries that we will consider.

Bart Vanluyten, Jan C. Willems and Bart De Moor are with the Electrical Engineering Department, K.U.Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium.

Bart Vanluyten is a Research Assistant with the fund for Scientific Research-Flanders (FWO-Vlaanderen).

A first example is the pointwise symmetry $PH(t)Q = H(t)$ for $t \in \mathbb{N}$. In words, the transformation Q applied to the inputs is compensated by the transformation P applied to the outputs. For example, we consider P and/or Q permutation matrices. This corresponds to systems in which some of the inputs and/or outputs can be interchanged, without changing the Markov parameters. Figure 1.a shows a system in which the outputs can be interchanged. Figure 1.b gives an example of a system in which the inputs can be interchanged. Another important case is when $Q = P^{-1}$ which occurs for example in systems with identical subsystems (Figure 2). Also of interest is the case in which P and/or Q are rotation matrices, etc.

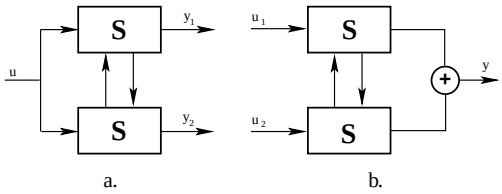

Fig. 1. Systems in which the outputs (subfigure a) or inputs (subfigure b) can be interchanged.

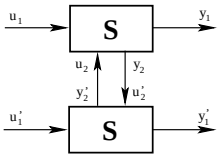

Fig. 2. System as an interconnection of two identical subsystems.

A second example has been studied in the interesting paper that stimulated us to study this problem [1]. It corresponds to systems with periodic impulse responses of period T, i.e.

$$H(t) = H(t+T),\ t \in \mathbb{N}.$$

We will also consider even, odd, or even/odd impulse responses.

In this paper we restrict ourselves to these two types of examples: pointwise symmetries and periodic impulse response symmetries. The problem to be considered is whether model reduction algorithms (e.g. balanced model reduction for the pointwise case) respects these symmetries.

III. SVD-REDUCTION OF MATRICES WITH SYMMETRIES

In this section, we prove an interesting property of the SVD-reduction of matrices. It will be the mathematical basis of our results on model reduction for dynamic systems. We

consider matrices over $\mathbb{R}$. A square matrix P is said to be [*unitary*] $:\Leftrightarrow$ [$P^\top P = I$]. The norm $||\cdot||$ on $\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be [*unitarily invariant*] $:\Leftrightarrow$

$$[(M \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}) \wedge (P,Q \text{ unitary})] \Rightarrow [||PMQ^\top|| = ||M||].$$

One example of a unitary invariant norm is the Frobenius norm. The Frobenius norm of $M = [m_{ij}]$ is defined as $||M||_{\mathfrak{F}} := \sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(m_{ij})^2}$.

Let $M \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$. Denote its singular values by $(\sigma_1(M),\sigma_2(M),\ldots,\sigma_{\min\{\mathtt{n}_1,\mathtt{n}_2\}}(M))$, ordered as

$$\sigma_1(M) \geq \sigma_2(M) \geq \ldots \geq \sigma_{\min\{\mathtt{n}_1,\mathtt{n}_2\}}(M).$$

Consider the Singular Value Decomposition (SVD) of M

$$M = U\Sigma V^\top,$$

with

$$\Sigma := \operatorname{diag}(\sigma_1(M),\sigma_2(M),\ldots,\sigma_{\min\{\mathtt{n}_1,\mathtt{n}_2\}}(M))$$

and $U \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ and $V \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{n}_2}$ unitary. Call

$$M_{\mathtt{k}} := U\begin{bmatrix}\Sigma_{\mathtt{k}} & 0\\ 0 & 0\end{bmatrix}V^\top$$

with

$$\Sigma_{\mathtt{k}} := \operatorname{diag}(\sigma_1(M),\sigma_2(M),\ldots,\sigma_{\mathtt{k}}(M))$$

the *rank* $\mathtt{k}$ *SVD-truncation* of M. It is well-known that, if the *gap condition*

$$\sigma_k(M) > \sigma_{\mathtt{k}+1}(M)$$

holds, then the rank $\mathtt{k}$ SVD-truncation of M is uniquely defined. Indeed, while the $\sigma(M)$'s are always uniquely defined, U and V are never unique, but nevertheless, if the gap condition holds, then the rank $\mathtt{k}$ SVD-truncation of M is unique.

The rank $\mathtt{k}$ SVD-truncation of M leads to an optimal rank $\mathtt{k}$ approximation of M, with respect to any unitarily matrix invariant norm. In other words

$$[(||\cdot|| \text{ unitarily invariant}) \wedge (\operatorname{rank}(M') \leq \mathtt{k})] \Rightarrow [||M-M'|| \geq ||M-M_{\mathtt{k}}||].$$

The purpose of this section is to prove a theorem concerning the preservation of a certain kind of symmetry after rank $\mathtt{k}$ SVD-truncation. It is based on the well-known fact that $M_{\mathtt{k}}$ is the *unique* matrix of rank $\mathtt{k}$ which approximates M optimally with respect to the Frobenius norm if the gap condition holds.

Proposition 1: If the gap condition $\sigma_k(M) > \sigma_{\mathtt{k}+1}(M)$ holds, then the rank $\mathtt{k}$ SVD-truncation $M_{\mathtt{k}}$ is the *unique* matrix of rank k which approximates M optimally in the Frobenius norm, i.e.

$$[\big(\sigma_k(M) > \sigma_{\mathtt{k}+1}(M)\big) \wedge (\operatorname{rank}(M'_{\mathtt{k}}) \leq \mathtt{k}) \wedge (||M-M'_{\mathtt{k}}||_{\mathfrak{F}} = ||M-M_{\mathtt{k}}||_{\mathfrak{F}})] \Rightarrow [M'_{\mathtt{k}} = M_{\mathtt{k}}]$$

Proof: This proposition is undoubtedly very well-known, but for the sake of completeness, we give a proof in appendix. ■

Of course, it follows that if the gap condition $\sigma_k(M) > \sigma_{\mathtt{k}+1}(M)$ holds, then the rank $\mathtt{k}$ SVD-truncation $M_{\mathtt{k}}$ is the *unique* matrix of rank $\mathtt{k}$ which approximates M optimally, simultaneously for all unitarily invariant norms. It is an interesting question for which unitarily invariant norms the analogue of Proposition 1 holds.

Using the above proposition, we are now able to prove the following theorem about the SVD of a matrix with symmetry.

Theorem 2: Assume that the matrix $M \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ has the following *symmetry*:

$$M = PMQ$$

with P and Q unitary matrices. Then, if

$$\sigma_{\mathtt{k}}(M) > \sigma_{\mathtt{k}+1}(M),$$

$M_{\mathtt{k}}$, the optimal rank $\mathtt{k}$ approximation derived from truncating the SVD, has the same symmetry:

$$M_{\mathtt{k}} = PM_{\mathtt{k}}Q.$$

Proof: The Frobenius norm is unitarily invariant, so

$$||M-M_{\mathtt{k}}||_{\mathfrak{F}} = ||P(M-M_{\mathtt{k}})Q||_{\mathfrak{F}} = ||M-PM_{\mathtt{k}}Q||_{\mathfrak{F}}.$$

Hence $PM_{\mathtt{k}}Q$ is an optimal rank $\mathtt{k}$ approximation of M with respect to the Frobenius norm. So by the uniqueness shown in Proposition 1, $PM_{\mathtt{k}}Q = M_{\mathtt{k}}$. ■

IV. EXAMPLES

In this section, we give some examples of matrices $M \in \mathbb{R}^{n_1\times\mathtt{n}_2}$ for which $M = PMQ$ with P and Q unitary matrices. We restrict the examples to matrices which are relevant for model reduction of LTI systems with symmetries.

A. Matrices with equal rows/columns

Let $P_{i,j}$ be the $\mathtt{n}_1 \times \mathtt{n}_1$ permutation matrix such that in $P_{i,j}x$ the i-th and j-th elements of x are permuted. Then in $P_{i,j}M$, the i-th and j-th rows are permuted. Now $M = P_{i,j}M$ means that the i-th and the j-rows of M are equal. Theorem 2 allows us to conclude that if the gap condition holds, then $M_{\mathtt{k}} = P_{i,j}M_{\mathtt{k}}$, i.e. the i-th and j-th rows of $M_{\mathtt{k}}$ are also equal. A matrix M for which the symmetry $M = P_{i,j}M$ holds for many pairs of (i,j), corresponds to either a matrix with more than two equal rows or a matrix with more than one group of rows which are identical. If the gap condition holds, all these symmetries separately are retained after SVD-truncation. Analogous results can be obtained for the columns of M.

B. Matrices with zero-rows/-columns

To express that the i-th row of M is zero, consider the matrix $P_i = \operatorname{diag}(1,\ldots,1,-1,1,\ldots,1)$, with the -1 on the i-th position, and express that $M = P_iM$. If the gap condition holds, then for the optimal rank $\mathtt{k}$ approximation of M holds that $M_{\mathtt{k}} = P_iM_{\mathtt{k}}$, i.e. the i-th row of $M_{\mathtt{k}}$ is also equal to zero. If the symmetry $M = P_iM$ holds for different values of i, then more than one row of M are equal to zero. All the symmetries separately are retained after SVD-truncation if the gap condition holds. Analogous results can be obtained for the columns of M.

C. Circulant matrices

In this section we consider block matrices with $\mathtt{n} \times \mathtt{n}$ blocks of size $\mathtt{p} \times \mathtt{m}$. Define the special permutation matrix $\Pi \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

$$\Pi = \begin{bmatrix} 0 & I_{\mathtt{n}-1} \\ 1 & 0 \end{bmatrix},$$

where $I_{\mathtt{n}-1}$ denotes the identity matrix of size $\mathtt{n}-1$. Let $F = \begin{bmatrix} F_1^\top & \dots & F_{\mathtt{n}}^\top \end{bmatrix}^\top$ with $F_i \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, $i = 1,\dots,\mathtt{n}$, then the block matrix $\mathscr{C}_F$ with $\mathtt{n}\times\mathtt{n}$ blocks of size $\mathtt{p}\times\mathtt{m}$

$$\mathscr{C}_F := \begin{bmatrix} F & (\Pi\otimes I_{\mathtt{p}})F & (\Pi\otimes I_{\mathtt{p}})^2F & \cdots & (\Pi\otimes I_{\mathtt{p}})^{\mathtt{n}-1}F \end{bmatrix}, \tag{1}$$

where $\otimes$ denotes the Kronecker product, is called the *block circulant matrix generated by* F. Such a matrix looks like

$$\mathscr{C}_F = \begin{bmatrix} F_1 & F_2 & \dots & F_{\mathtt{n}-1} & F_{\mathtt{n}} \\ F_2 & F_3 & \dots & F_{\mathtt{n}} & F_1 \\ \vdots & \vdots & & \vdots & \vdots \\ F_{\mathtt{n}-1} & F_{\mathtt{n}} & \dots & F_{\mathtt{n}-3} & F_{\mathtt{n}-2} \\ F_{\mathtt{n}} & F_1 & \dots & F_{\mathtt{n}-2} & F_{\mathtt{n}-1} \end{bmatrix}.$$

Observe the *block Hankel* structure of block circulant matrices. An equivalent way of defining block circulant matrices is:

$$[M \in \mathbb{R}^{\mathtt{np}\times\mathtt{nm}} \text{ is block circulant}] \Leftrightarrow [M = (\Pi\otimes I_{\mathtt{p}})M(\Pi\otimes I_{\mathtt{m}})].$$

A generalization of block circulant matrices are the *block* $\mathtt{g}$*-circulant matrices.* The block matrix $\mathscr{G}_F$ with $\mathtt{n}\times\mathtt{n}$ blocks of size $\mathtt{p}\times\mathtt{m}$ defined as

$$\mathscr{G}_F := \begin{bmatrix} F & (\Pi\otimes I_{\mathtt{p}})^{\mathtt{g}}F & (\Pi\otimes I_{\mathtt{p}})^{2\mathtt{g}}F & \cdots & (\Pi\otimes I_{\mathtt{p}})^{(\mathtt{n}-1)\mathtt{g}}F \end{bmatrix}$$

is called the block $\mathtt{g}$-circulant matrix generated by F. Again, an equivalent way of defining block $\mathtt{g}$-circulant matrices is:

$$[M \in \mathbb{R}^{\mathtt{np}\times\mathtt{nm}} \text{ is block } \mathtt{g}\text{-circulant}] \Leftrightarrow [M = (\Pi\otimes I_{\mathtt{p}})M(\Pi\otimes I_{\mathtt{p}})^{g}].$$

We already noticed that block circulant matrices have *block Hankel* structure. On the other hand a block $(\mathtt{n}-1)$-circulant matrix has *block Toeplitz* structure[1].

A second generalization of block circulant matrices are the *block skew-circulant matrices*. Define the special permutation-like matrix $\Theta \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

$$\Theta = \begin{bmatrix} 0 & I_{\mathtt{n}-1} \\ -1 & 0 \end{bmatrix}.$$

Let $F \in \mathbb{R}^{\mathtt{np}\times\mathtt{m}}$, then the block matrix $\mathscr{S}_a$ with $\mathtt{n}\times\mathtt{n}$ blocks of size $\mathtt{p}\times\mathtt{m}$

$$\mathscr{S}_F := \begin{bmatrix} F & (\Theta\otimes I_{\mathtt{p}}))F & (\Theta\otimes I_{\mathtt{p}})^2F & \cdots & (\Theta\otimes I_{\mathtt{p}})^{\mathtt{n}-1}F \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product, is called the block skew-circulant matrix generated by F. An equivalent way of defining block skew-circulant matrices is:

$$[M \in \mathbb{R}^{\mathtt{np}\times\mathtt{nm}} \text{ is block skew-circulant }] \Leftrightarrow [M = (\Theta\otimes I_{\mathtt{p}})M(\Theta\otimes I_{\mathtt{m}})].$$

It follows from Theorem 2 that if M is block circulant (in any of the senses considered above) and if the gap condition holds, then the truncated SVD $M_{\mathtt{k}}$ is also block circulant (in the same sense). We know from Proposition 1 that if the gap condition holds, the rank $\mathtt{k}$ SVD-truncation $M_{\mathtt{k}}$ is the unique matrix of rank $\mathtt{k}$ which approximates M optimally in the Frobenius norm. As a consequence of this, the SVD-truncation $M_{\mathtt{k}}$ of a block circulant matrix can very nicely be computed using the Discrete Fourier Transform (DFT). We explain this only for the vector case. Consider

$$M = \begin{bmatrix} m_1 & m_2 & \dots & m_{\mathtt{n}-1} & m_{\mathtt{n}} \\ m_2 & m_3 & \dots & m_{\mathtt{n}} & m_1 \\ \vdots & \vdots & & \vdots & \vdots \\ m_{\mathtt{n}-1} & m_{\mathtt{n}} & \dots & m_{\mathtt{n}-3} & m_{\mathtt{n}-2} \\ m_{\mathtt{n}} & m_1 & \dots & m_{\mathtt{n}-2} & m_{\mathtt{n}-1} \end{bmatrix},$$

with $m_t \in \mathbb{R}^{\mathtt{p}}$ for $t = 1,2,\dots n$. Let

$$\tilde{m}_{\mathtt{f}} := \sum_{t=1}^{\mathtt{n}} m_t e^{-i\mathtt{f}\frac{2\pi}{\mathtt{n}}t}, \quad \mathtt{f} = 0,1,\dots,\mathtt{n}-1$$

be the DFT of the first block row of M: $m_1, m_2, \dots, m_{\mathtt{n}}$, such that

$$m_t = \frac{1}{\mathtt{n}}\sum_{\mathtt{f}=0}^{\mathtt{n}-1} \tilde{m}_{\mathtt{f}} e^{i\mathtt{f}\frac{2\pi}{\mathtt{n}}t}, \quad t = 1,2,\dots,\mathtt{n}.$$

Using for example realization theory, it follows readily that the rank of M equals the cardinality of the set

$$\{\mathtt{f} \in \{0,1,\cdots,\mathtt{n}-1\} \mid ||\tilde{m}_{\mathtt{f}}|| \neq 0\}.$$

It is also known that

$$\begin{aligned} \frac{1}{\mathtt{n}}||M||_{\mathfrak{F}}^2 &= \sum_{t=1}^{\mathtt{n}} ||m_t||^2 \\ &= \frac{1}{\mathtt{n}}\sum_{f=0}^{\mathtt{n}-1} ||\tilde{m}_f||^2. \end{aligned}$$

Therefore, in order to obtain M_k, an optimal rank $\mathtt{k}$ approximation of M in the Frobenius norm, we can proceed as follows. First calculate

$$\hat{m}_t = \frac{1}{n}\sum_{\mathtt{f}\in F_{\mathtt{k}}} \tilde{m}_{\mathtt{f}} e^{i\mathtt{f}\frac{2\pi}{\mathtt{n}}t}, \quad t = 1,2,\dots,\mathtt{n},$$

with $F_{\mathtt{k}}$ the subset of $\{0,1,\dots,\mathtt{n}-1\}$ of cardinality $\mathtt{k}$ with the property

$$[(\mathtt{f} \in F_{\mathtt{k}}) \wedge (\mathtt{f}' \notin F_{\mathtt{k}})] \Rightarrow [||\tilde{m}_{\mathtt{f}}|| \ \geq || \ \tilde{m}_{\mathtt{f}'}||].$$

Now, it is easy to see that M_k is equal to the block circulant matrix induced by the vector $\begin{bmatrix} \hat{m}_1^\top & \hat{m}_2^\top & \dots & \hat{m}_n^\top \end{bmatrix}^\top$ (see equation (1)). Under obvious conditions on $F_{\mathtt{k}}$, $M_{\mathtt{k}}$ is real.

[1] Some authors define block circulant matrices to be block Toeplitz and their block $(\mathtt{n}-1)$-circulant matrices are block Hankel. For further use, we prefer the definition given above.

Note also that $M_{\mathtt{k}}$ is the unique optimal rank $\mathtt{k}$ approximation of M in the Frobenius norm if

$$[(\mathtt{f} \in F_{\mathtt{k}}) \wedge (\mathtt{f}' \notin F_{\mathtt{k}})] \Rightarrow [||\tilde{m}_{\mathtt{f}}|| \ > \ ||\tilde{m}_{\mathtt{f}'}||].$$

Assume that both these conditions are satisfied. Then $M_{\mathtt{k}}$ approximates M optimally in the Frobenius norm with a block circulant matrix of rank $\mathtt{k}$ and it is the unique block circulant matrix that does so. Hence we derived an alternative way to calculate the SVD-truncation $M_{\mathtt{k}}$ by making use of the DFT. Moreover, since $\hat{m}_t$, $t = 1,2,\ldots \mathtt{n}-1$ may be computed using the Fast Fourier Transform (FFT), it is numerically much more efficient to compute $M_{\mathtt{k}}$ by first computing $\hat{m}_t$, $t = 1,2,\ldots \mathtt{n}-1$ and then forming $M_{\mathtt{k}}$, than it is to compute the SVD. This observation is valid also when we look for an optimal rank $\mathtt{k}$ approximation of M in another unitarily invariant norm than the Frobenius norm.

V. APPLICATION TO MODEL REDUCTION

A. Impulse responses with pointwise symmetry

In this section, it is shown that if the Markov parameters $H(1), H(2), \ldots, H(t), \ldots$ of a stable (meaning $\sum_{t\in\mathbb{N}} ||H(t)|| < \infty$) system $\mathscr{S}$ have a pointwise symmetry, then the Markov parameters $H_{\text{red}}(1), H_{\text{red}}(2), \ldots$ of the balanced reduced system $\mathscr{S}_{\text{red}}$ have the same symmetry. We first prove this result and then present some applications.

Proposition 3: Assume that the system $\mathscr{S}$ is stable and that its Markov parameters have the symmetry

$$PH(t)Q = H(t), \quad t \in \mathbb{N},$$

with P and Q given unitary matrices. Then, if $\sigma_{\mathtt{k}}(\mathfrak{H}_H) > \sigma_{\mathtt{k}+1}(\mathfrak{H}_H)$, the Markov parameters of the balanced reduced system $\mathscr{S}_{\text{red}}$ of order $\mathtt{k}$ have the same symmetry:

$$PH_{\text{red}}(t)Q = H_{\text{red}}(t), \quad t \in \mathbb{N}.$$

Proof: A balanced realization of the system $\mathscr{S}$ can be obtained from the reduced SVD of its Hankel matrix $\mathfrak{H}_H = U\Sigma_H V^\top$ as

$$\begin{aligned} A &= \sqrt{\Sigma_H^{-1}}U^\top \mathfrak{H}_{\sigma H} V \sqrt{\Sigma_H^{-1}}, \\ B &= \sqrt{\Sigma_H^{-1}}U^\top \mathfrak{H}_H^{\infty,1}, \\ C &= \mathfrak{H}_H^{1,\infty} V \sqrt{\Sigma_H^{-1}}. \end{aligned}$$

where

$$\mathfrak{H}_{\sigma H} = \begin{bmatrix} H(2) & H(3) & H(4) & \cdots \\ H(3) & H(4) & H(5) & \cdots \\ H(4) & H(5) & H(6) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and $\mathfrak{H}_H^{i,j}$ denotes the submatrix of $\mathfrak{H}_H$ consisting of the first i block rows and j block columns. Express $\mathfrak{H}_H$ as

$$\mathfrak{H}_H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_{H_1} & 0 \\ 0 & \Sigma_{H_2} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^\top,$$

where the size of Σ_{H_1} is equal to $\mathtt{k}$. The balanced reduced system $\mathscr{S}_{\text{red}}$ of order $\mathtt{k}$ then has the realization

$$\begin{aligned} A_{\text{red}} &= \sqrt{\Sigma_{H_1}^{-1}}U_1^\top \mathfrak{H}_{\sigma H} V_1 \sqrt{\Sigma_{H_1}^{-1}}, \\ B_{\text{red}} &= \sqrt{\Sigma_{H_1}^{-1}}U_1^\top \mathfrak{H}_H^{\infty,1}, \\ C_{\text{red}} &= \mathfrak{H}_H^{1,\infty} V_1 \sqrt{\Sigma_{H_1}^{-1}}. \end{aligned}$$

Call $\mathscr{P} = I_\infty \otimes P$ and $\mathscr{Q} = I_\infty \otimes Q$, then

$$\mathfrak{H}_H = \mathscr{P}\mathfrak{H}_H\mathscr{Q}.$$

It follows from Theorem 2 that, if the condition $\sigma_{\mathtt{k}}(\mathfrak{H}_H) > \sigma_{\mathtt{k}+1}(\mathfrak{H}_H)$ holds,

$$\mathscr{P}U_1\Sigma_{H_1}V_1^\top\mathscr{Q} = U_1\Sigma_{H_1}V_1^\top.$$

Because the Moore-Penrose pseudo-inverse of a given matrix is uniquely defined, we also have that

$$\mathscr{Q}^\top V_1\Sigma_{H_1}^{-1}U_1^\top\mathscr{P}^\top = V_1\Sigma_{H_1}^{-1}U_1^\top.$$

The first Markov parameter is equal to

$$\begin{aligned} C_{\text{red}}B_{\text{red}} &= \mathfrak{H}_H^{1,\infty}V_1\Sigma_{H_1}^{-1}U_1^\top\mathfrak{H}_H^{\infty,1} \\ &= (P\mathfrak{H}_H^{1,\infty}\mathscr{Q})(\mathscr{Q}^\top V_1\Sigma_{H_1}^{-1}U_1^\top\mathscr{P}^\top)(\mathscr{P}\mathfrak{H}_H^{\infty,1}P) \\ &= PC_{\text{red}}B_{\text{red}}Q. \end{aligned}$$

The same can be done for $C_{\text{red}}A_{\text{red}}B_{\text{red}}$, $C_{\text{red}}A_{\text{red}}^2B_{\text{red}}, \ldots$ We conclude that

$$PC_{\text{red}}A_{\text{red}}^{t-1}B_{\text{red}}Q^\top = C_{\text{red}}A_{\text{red}}^{t-1}B_{\text{red}}, \quad t \in \mathbb{N}.$$

■

We now present some applications (assuming that the gap condition holds) of the above proposition.

Suppose that for all t, row i and j of the Markov parameters $H(t)$ of a system $\mathscr{S}$ are equal. In that case, we see that outputs y_i and y_j of the system $\mathscr{S}$ are identical. Now from Proposition 3, we know that the output $y_{\text{red},i}$ and output $y_{\text{red},j}$ of the balanced truncated system $\mathscr{S}_{\text{red}}$ are also equal.

Similarly, suppose that for all t, column i and j of the Markov parameters $H(t)$ of a system $\mathscr{S}$ are equal. In that case, we see that the output y of the system $\mathscr{S}$ does not depend on the i-th and j-input separately, but depends only on its sum. Now from Proposition 3, we know that the output y_{red} of the balanced truncated system $\mathscr{S}_{\text{red}}$ also depends only on the sum of inputs i and j.

If the the i-th column of $H(t)$ is equal to 0 for all t, the output of the system $\mathscr{S}$ does not depend on its i-th input. Again, we know from Proposition 3, that the output of $\mathscr{S}_{\text{red}}$ is also independent of the i-th input. Analogous conclusions can be drawn for the case where rows of $H(t)$ are equal to 0.

B. Periodic impulse response

Assume that the impulse response $H(1),H(2),\ldots,H(t),\ldots$ is periodic with period T: $H(t+T)=H(t)$ for $t\in\mathbb{N}$. The problem is to obtain a reduced order model with an impulse response which is also periodic. Now since $\text{rank}(\mathfrak{H}_H)=\text{rank}(\mathfrak{H}_H^{T,T})$, it is logical to look for a periodic H_{red} such that

$$||\mathfrak{H}_H^{T,T}-\mathfrak{H}_{H_{\text{red}}}^{T,T}||$$

is small and that $\text{rank}(\mathfrak{H}_{H_{\text{red}}}^{T,T})<\text{rank}(\mathfrak{H}_H^{T,T})$. Since $\mathfrak{H}_H^{T,T}$ is block circulant, the problem is to find a low rank block circulant approximation of a block circulant matrix. We know that if the gap condition holds, the truncated SVD of $\mathfrak{H}_H^{T,T}$ gives an optimal approximation in any unitarily invariant norm which is again block circulant. Moreover, it is shown in [1] that this reduction corresponds to reduction by finite time balancing. As was shown in the previous section, the SVD-truncation of the circulant Hankel matrix, can be efficiently computed using the DFT, which in addition can be implemented with the FFT-algorithm. This yields a fast way of computing a reduced order periodic model. This result is of relevance in image processing, as shown in [1] and [3].

C. Even/odd periodic impulse response

Assume that the impulse response $H(1),H(2),\ldots,H(t),\ldots$ is periodic with period T: $H(t+T)=H(t)$ for $t\in\mathbb{N}$. Consider in addition that the impulse response is *even*: $H(T-t)=H(t)$ for $t\in[1,T-1]$. The problem is to find a reduced order model with an impulse response which is also periodic and even. The Hankel matrix $\mathfrak{H}_H^{T,T}$ has two symmetries:

$$\begin{aligned}\mathfrak{H}_H^{T,T} &= (\Pi\otimes I_{\mathtt{p}})\mathfrak{H}_H^{T,T}(\Pi\otimes I_{\mathtt{m}})\\ \mathfrak{H}_H^{T,T} &= (\Lambda\otimes I_{\mathtt{p}})\mathfrak{H}_H^{T,T}(\Lambda\otimes I_{\mathtt{m}}),\end{aligned}$$

with

$$\Lambda=\begin{bmatrix} & & & 1\\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}.$$

If the gap condition holds, the truncated SVD of the Hankel matrix $\mathfrak{H}_H^{T,T}$ gives an optimal approximation in any unitarily invariant norm for which the same symmetries hold. Again, the problem can be solved more efficiently using DFT-techniques.

Analogous results can be obtained for an *odd* periodic impulse response $H(1),H(2),\ldots,H(t),\ldots$ with period T defined as: $H(t+T)=H(t)$ for $t\in\mathbb{N}$, $H(T-t)=-H(t)$ for $t\in[1,T-1]$ and $H(T)=0$. In that case, the Hankel matrix $\mathfrak{H}_H^{T,T}$ has the symmetries

$$\begin{aligned}\mathfrak{H}_H^{T,T} &= (\Pi\otimes I_{\mathtt{p}})\mathfrak{H}_H^{T,T}(\Pi\otimes I_{\mathtt{m}})\\ \mathfrak{H}_H^{T,T} &= (\Lambda\otimes I_{\mathtt{p}})\mathfrak{H}_H^{T,T}(-\Lambda\otimes I_{\mathtt{m}}).\end{aligned}$$

In the combination of the even and odd case, skew-circulant matrices pop up. For an *even-odd* periodic impulse response $H(1),H(2),\ldots,H(t),\ldots$ with period $2T$ it holds that: $H(t+2T)=H(t)$ for $t\in\mathbb{N}$, $H(T-t)=H(t)$ for $t\in[1,T-1]$, $H(2T-t)=-H(t)$ for $t\in[1,2T-1]$ and $H(T)=0$. In this case the Hankel matrix with the size equal to *half* the period, $\mathfrak{H}_H^{T,T}$ is block skew-circulant. The problem of finding a reduced order model with an impulse response which is also periodic and even-odd can again be solved by truncating the SVD of $\mathfrak{H}_H^{T,T}$.

VI. SIMULATION EXAMPLE

We consider the problem of how to model reduce a system consisting of the interconnection of many identical building blocks. Model reduction of interconnected systems while preserving the interconnection structure is important in many applications. In this section we study the interconnection of two identical building blocks shown in Figure 2. In order to model reduce the interconnected system, we can proceed in two ways: either model reduce the building block and interconnect, or model reduce the interconnected system and view the reduced model as an interconnection of identical subsystems. The simple simulations which we carried out showed that the second procedure gives much better results. Take a 'random' fourth order system for $\mathscr{S}$.

$$\begin{aligned}x(t+1) &= Ax(t)+\begin{bmatrix}B_1 & B_2\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}\\ \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} &= \begin{bmatrix}C_1\\ C_2\end{bmatrix}x(t),\end{aligned}\qquad(\mathscr{S})$$

with

$$A=\begin{bmatrix}-0.1067 & -0.1458 & -0.2499 & -0.0102\\ -0.2803 & -0.1569 & -0.0534 & 0.2273\\ 0.0680 & -0.0575 & -0.1349 & 0.2395\\ 0.0248 & 0.3294 & -0.0029 & -0.1033\end{bmatrix},$$

$$\begin{bmatrix}B_1 & B_2\end{bmatrix}=\begin{bmatrix}0.1209 & 1.1343\\ -0.2222 & 0\\ 0 & -1.4671\\ -0.3001 & 0\end{bmatrix},$$

$$\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0 & -0.6936 & -2.2374 & -0.0016\\ 0.5654 & 0.8339 & 0 & -1.6146\end{bmatrix}.$$

First order balanced reduction gives $\mathscr{S}_{\text{red}}$

$$\begin{aligned}A_{\text{red}} &= [-0.1322],\\ \begin{bmatrix}B_{1,\text{red}} & B_{2,\text{red}}\end{bmatrix} &= \begin{bmatrix}-0.1088 & -1.8262\end{bmatrix},\\ \begin{bmatrix}C_{1,\text{red}}\\ C_{2,\text{red}}\end{bmatrix} &= \begin{bmatrix}-1.7962\\ -0.3604\end{bmatrix}.\end{aligned}$$

The interconnected system is given by $\mathscr{S}_{\text{con}}$

$$\begin{aligned}A_{\text{con}} &= \begin{bmatrix}A & B_2C_2\\ B_2C_2 & A\end{bmatrix},\\ B_{\text{con}} &= \begin{bmatrix}B_1 & 0\\ 0 & B_1\end{bmatrix},\\ C_{\text{con}} &= \begin{bmatrix}C_1 & 0\\ 0 & C_1\end{bmatrix},\end{aligned}$$

which after second order balanced reduction gives $\mathscr{S}_{\text{con, red}}$

$$
\begin{aligned}
A_{\text{con, red}} &= \begin{bmatrix} -0.0647 & 0.8248 \\ 0.8248 & -0.0647 \end{bmatrix}, \\
B_{\text{con, red}} &= \begin{bmatrix} 0.8328 & -0.0075 \\ -0.0075 & 0.8328 \end{bmatrix}, \\
C_{\text{con, red}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Notice that $\mathscr{S}_{\text{con, red}}$ has the same symmetry as $\mathscr{S}_{\text{con}}$. After approximating $B_{\text{con, red}}$ by

$$
B_{\text{con, red}} \simeq \begin{bmatrix} 0.8328 & 0 \\ 0 & 0.8328 \end{bmatrix},
$$

$\mathscr{S}_{\text{con, red}}$ can be seen as the interconnection of two systems $\mathscr{S}'_{\text{red}}$

$$
\begin{aligned}
A'_{\text{red}} &= [-0.0647], \\
\begin{bmatrix} B'_{1,\text{red}} & B'_{2,\text{red}} \end{bmatrix} &= \begin{bmatrix} 0.8328 & \sqrt{0.8248} \end{bmatrix}, \\
\begin{bmatrix} C'_{1,\text{red}} \\ C'_{2,\text{red}} \end{bmatrix} &= \begin{bmatrix} 1 \\ \sqrt{0.8248} \end{bmatrix}.
\end{aligned}
$$

In Figure 3, we compare the impulse responses of

- the 8-th order interconnected system $\mathscr{S}_{\text{con}}$,
- the second order system obtained by interconnecting the first order approximations $\mathscr{S}_{\text{red}}$ of the building blocks,
- the second order system obtained by approximating the reduced interconnected system with an interconnection of two identical first order building blocks $\mathscr{S}'_{\text{red}}$.

It is clear from the figure, that the second approximation method, approximating the interconnected system and then viewing this reduction as an interconnection of the identical building blocks, yields the best results.

This simulation was inspired by Chapter 7 of [4].

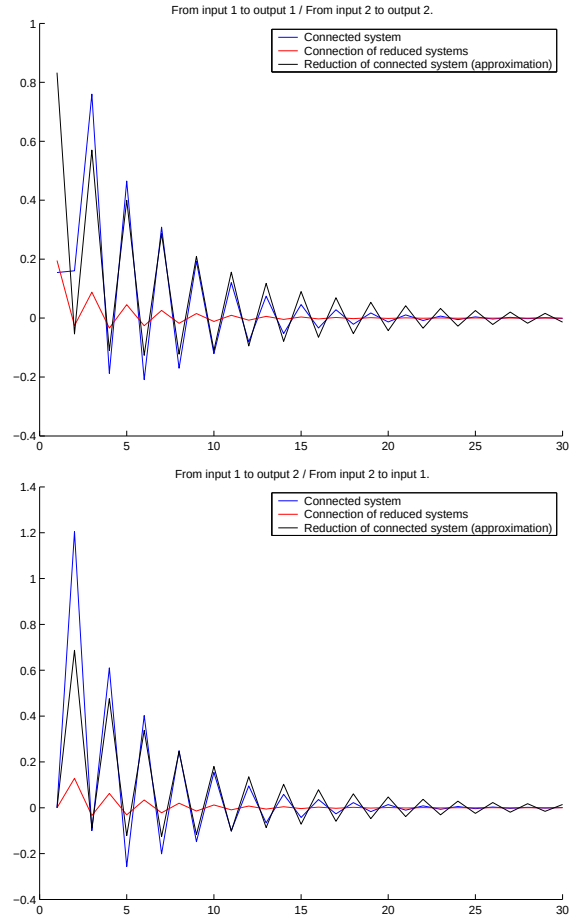

Fig. 3. Impulse responses (Above: from first input to first output 1, from second input to second output 2; Below: from first input to second output, from second output to first input) of a random 8-th order system (blue) $\mathscr{S}_{\text{con}}$, second order approximation of $\mathscr{S}_{\text{con}}$ obtained by interconnecting the first order approximation of its building blocks (red) and second order approximation of $\mathscr{S}_{\text{con}}$ obtained by approximating the reduced interconnected system with an interconnection of two identical first order building blocks (black).

VII. CONCLUSION

In this paper, we have shown how to model reduce LTI systems with pointwise symmetries and with periodic impulse responses. We have shown that model reduction based on SVD techniques preserves these symmetries if the 'gap condition' is satisfied. The results are based on the fact that the gap condition implies that the SVD-reduction of a matrix with unitary symmetries leads to a lower rank matrix with the same symmetries.

ACKNOWLEDGEMENTS

Our research is supported by:

Research Council KUL: GOA-Mefisto 666, GOA AM-BioRICS, several PhD/postdoc and fellow grants; Flemish Government: FWO: PhD/postdoc grants, projects, G.0240.99 (multilinear algebra), G.0407.02 (support vector machines), G.0197.02 (power islands), G.0141.03 (Identification and cryptography), G.0491.03 (control for intensive care glycemia), G.0120.03 (QIT), G.0452.04 (new quantum algorithms), G.0499.04 (Robust SVM), G.0499.04 (Statistics) research communities (ICCoS, ANMMM, MLDM); AWI: Bil. Int. Collaboration Hungary/ Poland; IWT: PhD Grants,GBOU (McKnow); Belgian Federal Science Policy Office: IUAP P5/22 ('Dynamical Systems and Control: Computation, Identification and Modelling', 2002-2006) ; PODO-II (CP/40: TMS and Sustainability); EU: FP5-Quprodis; ERNSI; Eureka 2063-IMPACT; Eureka 2419-FliTE; Contract Research/agreements: ISMC/IPCOS, Data4s, TML, Elia, LMS, Mastercard.

APPENDIX

A. Proof of Proposition 1

Let $M'_{\mathtt{k}}$ be an optimal rank $\mathtt{k}$ approximation of M, and let

$$
M'_k = U' \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix} V'^{\top}
$$

with $\Sigma'_{\mathtt{k}} \in \mathbb{R}^{\mathtt{k}\times\mathtt{k}}$, be an SVD of $M'_{\mathtt{k}}$. Then $\begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix}$ is obviously an optimal rank $\mathtt{k}$ approximation of $N := (U')^{\top} M V'$.

Partition

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

conformal with the partition $\begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix}$.

Observe that, since

$$\text{rank}(\begin{bmatrix} \Sigma'_{\mathtt{k}} & N_{12} \\ 0 & 0 \end{bmatrix} \leq \mathtt{k}$$

and

$$[N_{12} \neq 0] \Rightarrow [||N - \begin{bmatrix} \Sigma'_{\mathtt{k}} & N_{12} \\ 0 & 0 \end{bmatrix}||_{\mathfrak{F}} < ||N - \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix}||_{\mathfrak{F}}],$$

we obtain $N_{12} = 0$. Similarly, $N_{21} = 0$. Therefore $N = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{22} \end{bmatrix}$. Observe also that, since

$$\text{rank}(\begin{bmatrix} \Sigma'_{\mathtt{k}} - N_{11} & 0 \\ 0 & 0 \end{bmatrix} \leq \mathtt{k}$$

and

$$[N_{11} \neq \Sigma'_{\mathtt{k}}] \Rightarrow [||N - \begin{bmatrix} N_{11} & 0 \\ 0 & 0 \end{bmatrix}||_{\mathfrak{F}} < ||N - \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix}||_{\mathfrak{F}}],$$

we obtain $N_{11} = \Sigma'_{\mathtt{k}}$. Therefore $N = \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & N_{22} \end{bmatrix}$. Next, let $N_{22} = U_{22}\Sigma''_{\mathtt{k}}V_{22}^{\top}$ be an SVD of N_{22}, and note that

$$N' := \begin{bmatrix} I & 0 \\ 0 & U_{22}^{\top} \end{bmatrix} N \begin{bmatrix} I & 0 \\ 0 & V_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U_{22}^{\top} \end{bmatrix} (U')^{\top} M V' \begin{bmatrix} I & 0 \\ 0 & V_{22} \end{bmatrix}$$

is diagonal: $N' = \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & \Sigma''_{\mathtt{k}} \end{bmatrix}$, and has $\begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix}$ as an optimal rank $\mathtt{k}$ approximation. This obviously implies that the smallest diagonal element of $\Sigma'_{\mathtt{k}}$ is larger than the largest diagonal element of $\Sigma''_{\mathtt{k}}$. It follows that

$$M = U' \begin{bmatrix} I & 0 \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & \Sigma''_{\mathtt{k}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_{22}^{\top} \end{bmatrix} V'^{\top}$$

is an SVD of M and that

$$M'_{\mathtt{k}} = U' \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix} V'^{\top} = U' \begin{bmatrix} I & 0 \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma'_{\mathtt{k}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V_{22}^{\top} \end{bmatrix} V'^{\top}$$

is a rank $\mathtt{k}$ SVD-truncation of M.

Now, if the gap condition $\sigma_k(M) > \sigma_{\mathtt{k}+1}(M)$ holds, then the rank $\mathtt{k}$ SVD-truncation is unique. Hence $M'_{\mathtt{k}} = M_{\mathtt{k}}$. Conclude that $M_{\mathtt{k}}$ is then the *unique* optimal rank $\mathtt{k}$ approximation in the Frobenius norm of M.

REFERENCES

[1] M. Sznaier and O. Camps and C. Mazzaro, "Finite Horizon Model Reduction of a Class of Neutrally Stable Systems with Applications to Texture Synthesis and Recognition.", *Proceedings of the 43rd IEEE Conference on Decision and Control*, Bahamas, 2004, pp. 3068-3073.

[2] F. Fagnani and J.C. Willems, "Representations of Symmetric Linear Dynamical Systems", *SIAM Journal on Control and Optimization*, vol. 31, 1993, pp 1267-1293.

[3] C. Chang and A. Maciejewski and V. Balakrishnan, "Fast Eigenspace Decomposition of Correlated Images", *IEEE Transactions on Image Processing*, vol. 9, no. 11, 2000, pp 1937-1949.

[4] A. Vandendorpe, "Model Reduction of Linear Systems, an Interpolation Point of View", *PhD Thesis*, Faculté des sciences appliquées, Université catholique de Louvain, Belgium, 2004.