Direct Nonlinear Order Reduction with Variational Analysis*

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Abstract

The variational analysis [11] has been employed in [7] for order reduction of weakly nonlinear systems. For a relatively strong nonlinear system, this method will mostly lose efficiency because of the exponentially increased number of inputs in higher order variational equations caused by the individual reduction process of the variational systems. Moreover, the inexact inputs into the higher order variational equations indispensably introduce extra errors in the order reduction process. Inspired by the variational analysis, we propose a direct model order reduction method. The order of the approximate polynomial system of the original nonlinear system is directly reduced by one project space. The proposed direct reduction technique can easily avoid the errors brought by inexact inputs and the exponentially increased inputs. We show theoretically and experimentally that the proposed method can achieve much more accurate reduced system with smaller order size than the conventional variational equation order reduction method.

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1. Introduction

Model Order Reduction techniques have been widely applied in the fast simulation of large linear and nonlinear systems, such as IC interconnect circuits[1][2][3], high speed clock network [10], nonlinear analog RF circuits and MEMS systems [6][7][8][9] etc..

In the case of nonlinear systems, specific model order reduction techniques have been investigated recently. In the first category are the polynomial model reduction approaches [4][5][6]. In these methods, an approximate system is first obtained by Taylor expansion of the nonlinear term in the nonlinear system. Then either the linear part or some of the nonlinear parts in the Taylor expansion are employed to construct the projection space to reduce the order of the approximate system. For instance, the trajectory piece-wise linear method proposed in [4] approximates the nonlinear system by several linear systems piece-wisely. Each of the linear system is reduced by projection based linear order reduction method. The "quadratic reduction method" presented in [5] tries to approximate the nonlinear system by two order Taylor expansion to generate a quadratic system. However, the project space built by the linear part of the Taylor expansion hasn't preserved any nonlinear information. The bilinearization method in [6] derives the bilinear system in terms of Kronecker production of the state variables by the two order Taylor expansion of the nonlinear system. The reduction of the bilinear system is based on the Volterra series representation of bilinear system in control theory [11], which contains the nonlinear contribution in the multimoments. Therefore this method makes use of the nonlinear information of the original sys-

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tem not only in the bilinearization process but also in the reduction process.

The second category of nonlinear model order reduction techniques [7, 8] is based on the variational analysis theory in [11]. In this method, the original nonlinear system is transformed into several correlated linear systems, then model order reduction is performed on each linear system. Finally state variables of the nonlinear system are approximated by the linear combination of the state variables of the respective reduced linear systems. The conventional variational analysis based order reduction is simple for implementation since only several linear systems need to be reduced. However this method was found to be inefficient for relatively strong nonlinear systems, where the higher order variational equations are needed to ensure the reduction accuracy. In the later part of the paper, we will point out the number of the inputs of the higher order variational equation will grow exponentially, which makes it difficult to reduce the original system to a very small order size. Moreover the error induced in the reduction of the lower order variational equation will propagate to the higher order variational equations through the inputs and produce extra errors to the reduced system.

In this paper we propose a direct model order reduction method base on variational analysis. We show that it is much more efficient than the conventional variational equation order reduction methods [7, 8] in both reduction accuracy and the smaller order size of the reduced system. In Section 2, the principle of the conventional variational order reduction approach is reviewed and the limitations of this method are analyzed. In Section 3, we propose the direct variational order reduction technique. The efficiency of the proposed method is tested by circuit examples in Section 4. Finally we draw conclusions in Section 5.

2. Limitations of Variational Equation Model Order Reduction Method

In this section, we will at first review the variational equation model order reduction method and then analyze its limitations.

2.1. Review of variational equation model order reduction

The nonlinear systems that we are concerned with in this paper are of the following form.

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{b}u(t) \\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{x}(t) \end{aligned} \tag{1}$$

$\mathbf{x} \in \mathcal{R}^n$ is the state variables and the state-space dimension n denotes the order of the system. The initial condition is ($\mathbf{x}(0) = \mathbf{0}$). The input is denoted as $u = u(t)$ and the output response is $y = y(t)$. For simplicity, we only consider SISO (Single-Input Single-Output) system, where the input $u(t)$ and the output $y(t)$ are both scalar functions, therefore, $\mathbf{b} \in \mathcal{R}^n$, $\mathbf{c} \in \mathcal{R}^n$. We also have the assumption that $\mathbf{f}(\mathbf{x}(t))$ is smooth enough so that it can be expanded into Taylor series, which is the precondition for all the polynomial order reduction methods.

The variational equation approach is a method to derive the various kernels of a nonlinear system in control theory[11]. In [7][8] this method is used to perform order reduction on the nonlinear system given in (1). The detail of this method is presented as follows. Consider the response of (1) to the inputs of the form $\alpha u(t)$,

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{f}(\mathbf{x}) + \mathbf{b}(\alpha u) \\ y(t) &= \mathbf{c}^{\mathrm{T}}\mathbf{x}(t) \end{aligned} \tag{2}$$

where α is an arbitrary scalar. $\mathbf{x(t)}$ can be written as an expansion in the parameter α of the form:

$$\mathbf{x}(t) = \alpha\mathbf{x}_1(t) + \alpha^2\mathbf{x}_2(t) + \alpha^3\mathbf{x}_3(t) + \ldots \tag{3}$$

The Taylor series expansion of $\mathbf{f}(\mathbf{x})$ in the form of Kronecker product of the state variables is given in (4).

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{x}\otimes\mathbf{x} + \mathbf{A}_3\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x} + \cdots \tag{4}$$

Substituting (3) and (4) into the right hand side of (2), and substituting (3) into the left hand side of (2), we get

$$\begin{aligned} &\alpha\frac{d\mathbf{x}_1(t)}{dt} + \alpha^2\frac{d\mathbf{x}_2(t)}{dt} + \alpha^3\frac{d\mathbf{x}_3(t)}{dt} + \ldots \\ &= \alpha\mathbf{A}_1\mathbf{x}_1 + \alpha^2[\mathbf{A}_1\mathbf{x}_2 + \mathbf{A}_2(\mathbf{x}_1\otimes\mathbf{x}_1)] + \ldots + \mathbf{b}(\alpha u) \end{aligned} \tag{5}$$

Since this equation holds for all α, coefficients of like powers of α can be equated. This gives the variational equations:

$$\frac{d\mathbf{x}_1(t)}{dt} = \mathbf{A}_1\mathbf{x}_1 + \mathbf{b}u(t) \tag{6}$$

$$\frac{d\mathbf{x}_2(t)}{dt} = \mathbf{A}_1\mathbf{x}_2 + \mathbf{A}_2(\mathbf{x}_1\otimes\mathbf{x}_1) \tag{7}$$

$$\frac{d\mathbf{x}_3(t)}{dt} = \mathbf{A}_1\mathbf{x}_3 + \mathbf{A}_2(\mathbf{x}_1\otimes\mathbf{x}_2 + \mathbf{x}_2\otimes\mathbf{x}_1) + \mathbf{A}_3(\mathbf{x}_1\otimes\mathbf{x}_1\otimes\mathbf{x}_1) \tag{8}$$

$$\vdots$$

The idea of this method is that instead of reducing the nonlinear system (1), we only need to reduce the above linear systems (6), (7) and (8). Then $\mathbf{x}(t)$ can be gotten through $\mathbf{x}_k(t)$, $k = 1, 2, 3, \ldots$ by (3) and the response $y(t)$ is also obtained.

In [7, 8], projection method is used to reduce the order of the above linear systems. For example, for the first linear system (6), a projection matrix V_1 is computed based on

$\mathbf{A}_1$, $\mathbf{b}$, such that the columns of V_1 span the Krylov subspace $K_{q_1}(\mathbf{A}_1^{-1}, \mathbf{A}_1^{-1}\mathbf{b})$, i.e.

$$\begin{aligned} spancolumn\{V_1\} &= K_{q_1}(\mathbf{A}_1^{-1}, \mathbf{A}_1^{-1}\mathbf{b}) \\ &\equiv span\{\mathbf{A}_1^{-1}\mathbf{b}, \mathbf{A}_1^{-2}\mathbf{b}, ..., \mathbf{A}_1^{-q_1}\mathbf{b}\} \end{aligned} \tag{9}$$

For a single input system (6), the number of the columns in V_1 is usually q_1. Therefore, through variable change $\mathbf{x}_1 \approx V_1\mathbf{z}_1$, we have $z_1 \in R^{q_1}$, and the reduced linear system of (6) is obtained in (10),

$$\begin{aligned} \frac{d\mathbf{z}_1}{dt} &= \tilde{\mathbf{A}}_1\mathbf{z}_1 + \tilde{\mathbf{b}}u(t) \\ \tilde{y}_1 &= \tilde{\mathbf{c}}^{\mathrm{T}}\mathbf{z}_1 \end{aligned} \tag{10}$$

where $\tilde{\mathbf{A}}_1 = (V_1^{\mathrm{T}}\mathbf{A}_1^{-1}V_1)^{-1}$, $\tilde{\mathbf{b}} = \tilde{\mathbf{A}}_1 V_1^{\mathrm{T}}\mathbf{A}_1^{-1}\mathbf{b}$, $\tilde{\mathbf{c}} = V_1^{\mathrm{T}}\mathbf{c}$.

For this reduced system, the following theorem is well known in model order reduction of linear systems [6, 7].

Theorem 1 *The first q_1 moments of the reduced transfer function $\tilde{H}(s) = -\tilde{\mathbf{c}}^{\mathrm{T}}(\mathbf{I} - s\tilde{\mathbf{A}}_1^{-1})^{-1}\tilde{\mathbf{A}}_1^{-1}\tilde{\mathbf{b}}$ of the system in (10) are the same as those of the transfer function $H(s) = -\mathbf{c}^{\mathrm{T}}(\mathbf{I} - s\mathbf{A}_1^{-1})^{-1}\mathbf{A}_1^{-1}\mathbf{b}$ of the original linear system in (6).*

From Theorem 1, we see that the precision of the reduced model of (6) is directly decided by its order q_1. If q_1 is small, the precision of the reduced system may not be high enough because only small number of moments of the original transfer function are matched.

2.2. Limitations of variational method

2.2.1. Exponentially increased number of inputs

In this subsection, we will show that the numbers of the inputs of the second system (7) and third system (8) grow exponentially with respect to the order q_1 of the first reduced system (10). The number of inputs of the second linear system (7) can be derived from (7) and (10).

$$\mathbf{x}_1 \otimes \mathbf{x}_1 \approx V_1\mathbf{z}_1 \otimes V_1\mathbf{z}_1 = (V_1 \otimes V_1)(\mathbf{z}_1 \otimes \mathbf{z}_1)$$

Clearly, the number of inputs in the second linear system (7) becomes q_1^2.

For relatively strong nonlinear systems, order reduction of the third linear system (8) is necessary to achieve a reasonable accurate output response. This necessity is demonstrated by the circuit experiments in Figure 2 of Section 4.1. The third linear system (8) contains the term $(\mathbf{z}_1 \otimes \mathbf{z}_1 \otimes \mathbf{z}_1)$ which denotes the number of the inputs in (8) is at least q_1^3. Although the dimension of the inputs in (8) is decreased from n^3 to q_1^3, it is still a large number if q_1 is not small enough.

So far the problem of the conventional variational equation reduction method is clear. On the one hand if q_1 is not small enough, the input number q_1^3 will be large, then it is difficulty to reduce the order of the third linear system (8) to a small size. On the other hand if q_1 is small enough to reduce the order of (8) to a reasonable small size, the error of the reduced model will be unfortunately too large to be accepted because only very small number of moments of the original transfer function are matched. We illustrate this problem with experimental results line(e) and line(f) in Figure 2 in Section 4.1.

2.2.2. Errors caused by inexact inputs

Besides the problem of large number of inputs, the errors caused by the order reduction of the lower degree linear systems will be propagated to the higher degree linear systems through the inputs. For example, when the order of the first linear system (6) is reduced, the inputs $\mathbf{x}_1 \otimes \mathbf{x}_1$ into the second linear system (7) will be approximated by $\mathbf{z}_1 \otimes \mathbf{z}_1$, which means the actual inputs into the second linear system (7) are not exact. These errors in the inputs will induce extra errors in the second system (7) and will also propagate to the third system (8). In section 4.2, experimental results in Table 1 have shown that the errors introduced by the inexact inputs are even larger than the errors produced by the projection reduction of the linear systems.

3. Direct Order Reduction Method with Variational Analysis

In this section we propose the direct order reduction method for nonlinear systems to tackle with the problems of inexact inputs and increased number of inputs in the conventional variational order reduction method.

At first, with the Taylor expansion of $\mathbf{f}(\mathbf{x})$ in (4), we approximate the original nonlinear system (1) into a second order polynomial system in (11) or a third order polynomial system in (12).

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x} \otimes \mathbf{x}) + \mathbf{b}u \\ y(t) &= \mathbf{c}^{\mathrm{T}}\mathbf{x} \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x} \otimes \mathbf{x}) + \mathbf{A}_3(\mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x}) + \mathbf{b}u \\ y(t) &= \mathbf{c}^{\mathrm{T}}\mathbf{x} \end{aligned} \tag{12}$$

From the induction of (3) to (8), when $\alpha = 1$, it is easy to see that the solution of (11) is equivalent to

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) \tag{13}$$

and the solution of (12) is equivalent to

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) + \mathbf{x}_3(t) \tag{14}$$

Instead of reducing the individual linear systems (6,7,8) by individual projection sub-spaces, we develop a single projection matrix V to reduce the order of the whole system in (11) or (12).

Based on the moment matching projection order reduction methods for linear systems, we first construct a projection matrix V_1 based on the first linear system (6) such that

$$spancolum\{V_1\} = span\{\mathbf{A}_1^{-1}\mathbf{b}, \mathbf{A}_1^{-2}\mathbf{b}, \cdots, \mathbf{A}_1^{-q_1}\mathbf{b}\}.$$

Then we approximate $\mathbf{x}_1$ by $V_1\mathbf{z}_1$, i.e. $\mathbf{x}_1 \approx V_1\mathbf{z}_1$. Similarly, we generate a projection matrix V_2 based on the second linear system (7) such that

$$spancolum\{V_2\} = span\{\mathbf{A}_1^{-1}\mathbf{A}_2, \mathbf{A}_1^{-2}\mathbf{A}_2, \cdots \mathbf{A}_1^{-q_2}\mathbf{A}_2\}.$$

We also make approximation of $\mathbf{x}_2 \approx V_2\mathbf{z}_2$. For relatively strong nonlinear systems, if high reduction accuracy is desired, a projection matrix V_3 of the third linear system (8) needs to be derived in a similar way as the construction of V_2 [8]. From (14), we have

$$\mathbf{x}(t) \approx V_1\mathbf{z}_1 + V_2\mathbf{z}_2 + V_3\mathbf{z}_3$$

which indicates that the solution $\mathbf{x}(t)$ of (12) can be approximated by the linear combination of the column vectors in V_1, V_2 and V_3. Then the final orthogonal projection matrix V is developed by

$$spancolum\{V\} = spancolum\{V_1, V_2, V_3\}.$$

The approximate polynomial system (12) is thus reduced by $\mathbf{x} \approx V\mathbf{z}$ and multiplying with V^{T} on both sides of (12).

$$\begin{aligned} \frac{d\mathbf{z}(t)}{dt} &= V^{\mathrm{T}}\mathbf{A}_1V\mathbf{z} + V^{\mathrm{T}}\mathbf{A}_2(V\mathbf{z}\otimes V\mathbf{z}) \\ &\quad + V^{\mathrm{T}}\mathbf{A}_3(V\mathbf{z}\otimes V\mathbf{z}\otimes V\mathbf{z}) + V^{\mathrm{T}}\mathbf{b}u \\ y(t) &= \mathbf{c}^{\mathrm{T}}V\mathbf{z} \end{aligned} \qquad (15)$$

The obtained system (15) can be considered as the reduced system of the original nonlinear system (1).

From the above analysis, we remark that the direct projection variational method is more efficient than the conventional variational method in three aspects. At first, the inputs in the direct method are always the original inputs. Therefore the direct method can avoid the exponentially increased dimension of inputs and achieve even lower order reduced system than the conventional variational method. Secondly, the direct projection is more accurate than the conventional variational method because direct projection avoids the errors brought by the inexact inputs $\mathbf{z}_1 \otimes \mathbf{z}_1$ in (7), probably also the inexact inputs $\mathbf{z}_1 \otimes \mathbf{z}_1 \otimes \mathbf{z}_1$ in (8). Thirdly, the computation cost of the direct reduction is also saved because only one reduced system such as (15) needs to be solved. However, in variational equation order reduction method two or three reduced systems have to be solved.

In the following section, we demonstrate the efficiency of the proposed direct projection method by several circuit experiments.

4. Experiment Results

In this paper, the nonlinear circuit in [5] is employed as the test example. In the circuit, there are total N nodes, where we assume $N = 100$. There is one input current source $i(t) = u(t)$ flowing into node 1. The output response is set to be the voltage at node 1.

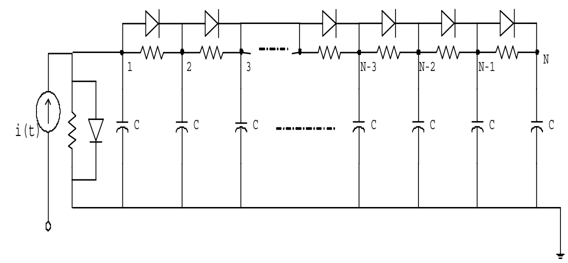

Figure 1. Circuit example

In the following, we use the solution of (1) computed by Matlab function as the exact response of the original system during the comparison. We reduce the three linear systems (6)(7)(8) to the order of q_1, q_2, q_3 and match the number of $j_1,\ j_2,\ j_3$ moments of the transfer functions of systems (6)(7)(8) respectively. We employ four different input signals to test the accuracy of the reduced model to the original nonlinear system (1). The four inputs are a step function, $u_1 = 0, when\ t \leq 3, else\ u_1 = 1,\ t \in [0, 10]$; an exponential function $u_2 = -e^{-t},\ t \in [0, 10]$; a cosine function $u_3 = (cos(2\pi t/10) + 1)/2,\ t \in [0, 10]$ and a sine function $u_4 = 1 + sin(2\pi t) + sin(10\pi t)$, $t \in [0, 1]$.

4.1. Exponentially increased number of inputs

In this subsection, we first illustrate that for relatively strong nonlinear systems, the third linear system (8) is necessary to achieve a reasonably accurate reduced system in the conventional variational model order reduction method. We will further demonstrate that due to the exponentially increased number of inputs in the second and third systems (7) (8), the order of the reduced system may not be small enough if high reduction accuracy is required. In Figure 2, the exact output response of the original nonlinear system is shown in the solid line (c). The exact output response of the first two linear systems (6)(7) is shown in line (d). Comparing line (c) and (d), one can see that even without order reduction, approximating the original system by only two linear systems (6)(7) will cause a definite error. If we further reduce the order of the first two linear systems(6)(7), even larger error will be generated as shown in line (a), where the first linear system is reduced to order $q_1 = 12$ and $j_1 = 12$ moments are matched, while the second linear system is reduced to order $q_2 = 22$ and only one ($j_2 = 1$) moment is matched. We can see that even if we increase the order of the second system, the order reduction accuracy can not be improved as shown in line (b), where we reduce the second linear system to order $q_2 = 32$ and match five ($j_2 = 5$) moments. This experiment implicates that the third linear system (8) is needed to enhance the accuracy of the order re-

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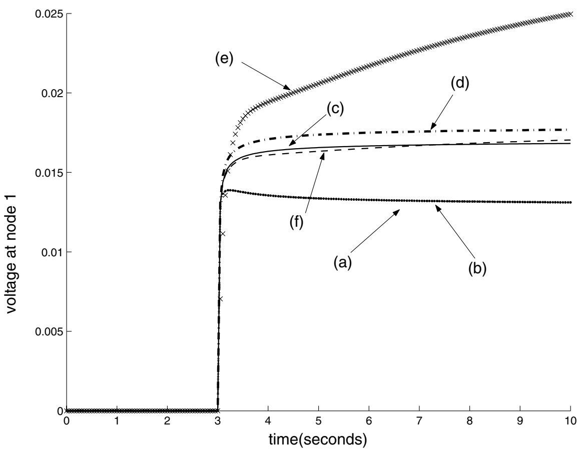

Figure 2. (a) Output response of the variational equation reduction method with $j_1 = 12, q_1 = 12$; $j_2 = 1, q_2 = 22$. (b) Output response of the variational equation reduction method with $j_1 = 12, q_1 = 12$; $j_2 = 5, q_2 = 32$. (c) Original output response. (d) Output response of the first two linear systems without order reduction. (e) Output response of the variational equation reduction method with $j_1 = 6, q_1 = 6$; $j_2 = 1, q_2 = 10$; $j_3 = 1, q_3 = 28$. (f) Output response of the variational equation reduction method with $j_1 = 12, q_1 = 12$; $j_2 = 1, q_2 = 22$; $j_3 = 1, q_3 = 65$. Input signal: $u_1(t)1 = 0,\ t <= 3;\ else\ u_1(t) = 1,\ 10 > t > 3$.

duction.

We further demonstrate the problem of large number of inputs when the third linear system is included in order reduction. At first, we reduce the first linear system (6) to the order of $q_1 = 6$ and match $j_1 = 6$ moments. For the reduction of the second linear system (7), if we only match the first moment (i.e. $j_2 = 1$), we get a reduced system of order $q_2 = 10$. In order to reduce the third linear system to a moderate size, we only match the first moment and obtain a reduced order $q_3 = 28$. The resulting output response of this reduction is shown in the crossing line (e) in Figure 2, which is far away from the exact solution in the solid line (c). On the other hand, if a more precise reduction is required, we must increase the order of the first system q_1. For example, if we increase q_1 from $q_1 = 6$ to $q_1 = 12$, we have to reduce the second linear system to $q_2 = 22$ and the third linear system to $q_3 = 65$, where only the first moments of the transfer functions of (7) and (8) are matched respectively. Now the accuracy of this reduction is much higher as shown in the dashed line (f) in Figure 2. However, the order of the reduced system is $q_3 = 65$, which is much close to the original system order $n = 100$. Clearly it is not a really reduced system, which makes the whole order reduction process much more inefficient. In the end, we point out that the problems of large dimension of inputs and inexact inputs still exist for other testing input signals such as u_2, u_3, u_4. Due to the space limitation, we omit the experiment results here.

	u_1	u_2	u_3	u_4
$error_1$	0.0012	0.0049	0.0017	0.0173
$error_2$	0.0014	0.0100	0.0024	0.0201

Table 1. Comparison of the inexact input errors with projection reduction errors.

4.2. Errors caused by inexact inputs

In the following, we show how much error induced by the inexact inputs $\mathbf{z}_1 \otimes \mathbf{z}_1$.

We first compute the error caused by the reduction of the first linear system (6). Denote $y_1 = \mathbf{c}^T \mathbf{x}_1$ the exact output response of system (6) without reduction. Denote $\tilde{y}_1 = \mathbf{c}^T (V_1 \mathbf{z}_1)$ the approximate output response of the first linear system computed through its reduced system in (16). Denote $error_1 = \|\tilde{y}_1 - y_1\|_2 / \|y_1\|_2$ the relative error between $\tilde{y}_1$ and y_1, i.e., the error caused by the projection reduction of the first linear system (6). Here, we reduce the first linear system to the order of $q_1 = 12$.

$$\frac{d\mathbf{z}_1}{dt} = \tilde{\mathbf{A}}_1 \mathbf{z}_1 + \tilde{\mathbf{b}} u(t) \tag{16}$$

Secondly, we compute the error induced only by the inexact input $(\mathbf{z}_1 \otimes \mathbf{z}_1)$ in (17). Denote $y_2 = \mathbf{c}^T \mathbf{x}_2$ the exact output response of system (7) without reduction. Denote $\hat{y}_2 = \mathbf{c}^T \hat{\mathbf{x}}_2$ the approximate output response in (17). Denote $error_2 = \|\hat{y}_2 - y_2\|_2 / \|y_2\|_2$ the relative error between $\hat{y}_2$ and y_2, i.e., the error induced only by the inexact input $(\mathbf{z}_1 \otimes \mathbf{z}_1)$ in (17).

$$\frac{d\hat{\mathbf{x}}_2(t)}{dt} = \mathbf{A}_1 \hat{\mathbf{x}}_2 + \mathbf{A}_2 (V_1 \mathbf{z}_1 \otimes V_1 \mathbf{z}_1) \tag{17}$$

The errors $error_1$, $error_2$ for the four different input signals u_1, u_2, u_3, u_4 are listed in Table 1. We can see that the error $error_2$ brought by the inexact inputs $(\mathbf{z}_1 \otimes \mathbf{z}_1)$ is even larger than the error induced by reduction projection $\mathbf{x}_1 \approx V_1 \mathbf{z}_1$ ($error_1$). Fortunately, in the direct project reduction method, there are no inexact inputs, therefore the inexact input error $error_2$ is easily avoided.

	u_1	u_2	u_3	u_4
ε_1	0.0071	0.0229	0.0049	0.0183
ε_2	0.0097	0.0437	0.0204	0.0912

Table 2. Comparison of the errors between direct reduction and the conventional variational reduction.

4.3. Accuracy of the direct reduction method

Finally, we compare the direct reduction method with the conventional variational reduction method. As shown in Figure 3, the response of the reduced system by the direct reduction method with order 10 in line (o) are more accurate than that by the conventional variational order reduction with order 65 in line (e) with testing input signal u_1.

The 65th order and the 10th order reduced systems are further tested with the other three testing inputs u_2, u_3 and u_4. Compared to the exact response of the original system, the relative error ε_1 of the direct reduction method and the relative error ε_2 of the conventional variational order reduction method are listed in Table 2. We can see that the direct reduction method is more accurate than the conventional variational method in all of the test inputs.

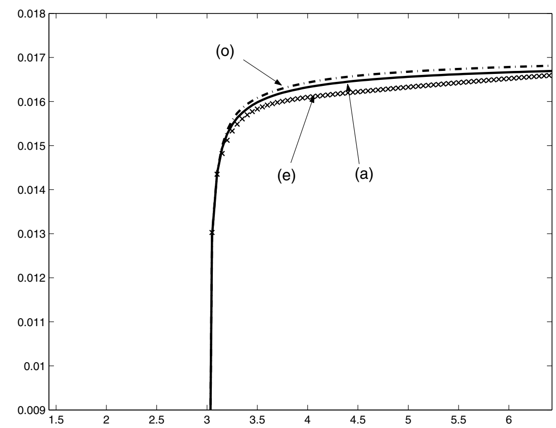

Figure 3. (a) Original output response; (e) Output response of the variational equation reduction method with $j_1 = 12, q_1 = 12;\ j_2 = 1, q_2 = 22;\ j_3 = 1, q_3 = 65$**; (o) Output response of the direct order reduction method with reduced order** $q = 10$**. Input signal:** $u_1(t) = 0,\ t <= 3;\ else\ u_1(t) = 1$**.**

5. Conclusion

For relatively strong nonlinear systems, the conventional variational order reduction method will lose efficiency due to the inexact input errors and the exponentially increased number of inputs. Based on the variational analysis principle, we propose in this paper a direct method to reduce the order of the approximate polynomial system by only one projection space. Both theoretical and experiment results have demonstrated the efficiency of the proposed direct projection method in both reduction accuracy and the smaller size of the reduced order system. Although the direct reduction method is derived for the time invariant nonlinear system, it definitely can be employed to deal with the order reduction of nonlinear time varying systems.

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