Perturbation analysis of TBR model reduction in application to trajectory-piecewise linear algorithm for MEMS structures.

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ABSTRACT

In this paper we use perturbation theory approach to analyze using truncated balanced realization (TBR) linear reduction in a Trajectory-Piecewise linear (TPWL) nonlinear model reduction method. We show that the most important factor affecting perturbation properties of the reduction basis of TBR is a spacing of Hankel singular values. The result is applied to choosing an order of reduction basis.

Keywords: model order reduction, nonlinear systems, truncated balanced realization

1 INTRODUCTION

MEMS devices generate challenging test cases for nonlinear model order reduction methods, due to their strongly nonlinear behavior. One of the model order reduction methods which can handle this nonlinear behavior is a Trajectory-Piecewise linear model order reduction (TPWL MOR) algorithm [1]. In our previous paper [2] we addressed one of the most important questions for TPWL MOR, namely the choice of linear reduction procedures. In the above mentioned paper we have showed that a truncated-balanced realization (TBR) linear reduction produces much more accurate reduced TPWL models than ones in which the Krylov-subspace linear reduction methods were used. However, for the case of a micromachined switch example (figure 1), we observed unexpected behavior. Some of the reduced models were very accurate, but some were unstable. This observation raised a totally new question - how can we choose a correct order for linear reduction in such way as to account for perturbations in linear model caused by nonlinearity?

The paper is organized as follows. In the section 2 we summarize trajectory piecewise-linear model order reduction for nonlinear systems, in section 3 we describe the TBR linear reduction procedure. In section 4 we analyze the TBR reduction algorithm from the perturbation point of view, which is the main contribution of this paper. In the next section we illustrate our insights with computational results for the case of micromachined switch reduced simulation.

2 TRAJECTORY PIECEWISE LINEAR MACROMODELS

In this paper we consider a class of nonlinear dynamical systems which can be represented using the standard state space form:

$$\begin{cases} \dot{x}(t) = f(x(t)) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in R^N$ is a vector of states at time t, $f : R^N \to R^N$ is a nonlinear vector-valued function, B is an $N \times M$ input matrix, $u : R \to R^M$ is an input signal, C is an $N \times K$ output matrix and $y : R \to R^K$ is the output signal.

In the context of nonlinear systems, the ultimate goal of model order reduction techniques is constructing macromodels capable of approximately simulating the input-output behavior of systems in form (1) at a significantly reduced numerical cost. We are using the following quasi-piecewise-linear approximate representation of the nonlinear function f, which has been proposed in [1]:

$$f(x) \approx \sum_{i=0}^{s-1} \tilde{w}_i(x) \left(f(x_i) + A_i(x - x_i) \right), \tag{2}$$

where x_i's ($i = 1, \ldots, (s-1)$) are some linearization points (states), A_i are the Jacobians of f evaluated at states x_i, and $\tilde{w}_i(x)$'s are state-dependent weights ($\sum_{i=0}^{s-1} \tilde{w}_i(x) = 1$ for all x). Applying the above approximation, and performing a projection of system (1) with biorthonormal matrices V and W yields:

$$\begin{cases} \dot{z} = (\sum_{i=0}^{s-1} w_i(z) A_{ir}) z + \gamma \cdot w(z) + B_r u \\ y = C_r z \end{cases}, \tag{3}$$

where:

$$\gamma = \left[W^T(f(x_0) - A_0 x_0), \ldots, W^T(f(x_{s-1}) - A_{s-1} x_{s-1}) \right],$$

$w(z) = [w_0(z) \ldots w_{s-1}(z)]^T$ ($\sum_{i=0}^{s-1} w_i(z) = 1$ for all z) is a vector of weights, $A_{ir} = W^T A_i V$, $B_r = W^T B$, and $C_r = CV$. One should note that evaluation of the right hand side of equation (3) requires at most $O(sq^2)$ operations, where s is the number of linearization points used.

As proposed in [1], linearization points x_i used in system (3) are picked from a 'training trajectory' of the initial nonlinear system, corresponding to some appropriately selected 'training input'.

In order to obtain a reduced system in form (3) one also needs to pick suitable biorthonormal projection bases V and W. This issue is addressed in more detail in the following section.

3 GENERATION OF THE REDUCTION BASIS

In the simplest projection strategy we consider linearization of (1) at the initial state x_0:

$$\begin{cases} \dot{x} = \hat{A}x + f(x_0) - \hat{A}x_0 + \hat{B}u \\ y = \hat{C}x \end{cases} \quad (4)$$

The projection basis can be obtained using any linear MOR procedure. To this end, the following square-root TBR procedure [3] is applied:

TBR($\hat{A}, \hat{B}, \hat{C}$)
Input: System matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$.
Output: Projection bases V and W.
(1) Find observability Gramian P:
$\hat{A}P + P\hat{A}^T = -\hat{B}\hat{B}^T$;
(2) Find controllability Gramian Q:
$\hat{A}^TQ + Q\hat{A} = -\hat{C}^T\hat{C}$;
(3) Using eigenvalue decomposition, compute Cholesky factors of P and Q:
$P = Z_cZ_c^T, \quad Q = Z_oZ_o^T$;
(4) Compute SVD of Cholesky product: $U\Sigma V = Z_o^TZ_c$;
(5) Compute V and W:
$V = Z_cVT\Sigma^{-1/2}, \quad W = Z_oUT\Sigma^{-1/2}$;
where $T = [I_{(q\times q)}0]^T$ is an $N \times q$ truncation matrix.

The obtained projection bases V and W are then used to compute the reduced order Jacobians A_{ir} (cf. (3)).

As an extended approach, we can also combine several projection bases from different linearization points and then aggregate them into one single basis using a biorthogonalization algorithm.

4 PERTURBATION ANALYSIS FOR TBR-BASED MACROMODELS

Assume the projection bases V and W are computed using TBR reduction, obtained at a single linearization point. The analysis can readily be extended to the case where we use aggregation of bases from different linearization points.

The key issue is whether or not the TBR basis obtained at one linearization point is still suitable for obtaining piecewise-linear reduced models further along the trajectory. To understand this issue, consider two non-reduced linear models (A, B, C) (initial) and $(\tilde{A}, B, C)$ (perturbed). Suppose TBR reduction is used on both of these models, resulting in projection bases V, W and $\tilde{V}, \tilde{W}$ respectively. We can say that if these bases are not significantly different, we can still use V and W for reducing the perturbed system.

Next, consider how perturbation in the system affects the projection basis for TBR reduction algorithm.

4.1 Effect of Perturbation on Gramians

We shall perform calculations for controllability gramian P only, the results are valid for Q as well. Given $A = A_0 + \delta A$, $P = P_0 + \delta P$, where P_0 is an unperturbed gramian corresponding to unperturbed matrix A_0. We assume δA is relatively small such that δP is also small.

Now plug in perturbed values of A and P into the Lyapunov equation. Note that term $\delta P\delta A$ may be neglected (since it is a second order term). We get:

$$A_0\delta P + \delta PA_0^T + (\delta AP_0 + P_0(\delta A)^T) = 0. \quad (5)$$

This is a Lyapunov equation with the same matrix A_0 as for unperturbed system. This equation has a unique solution, since we assume initial system to be stable. The solution to (5) satisfies the following equation:

$$\delta P = \int_0^\infty e^{A_0^Tt}(\delta AP_0 + P_0(\delta A)^T)e^{A_0t}dt. \quad (6)$$

Assuming A is diagonalizable, we can obtain the following bound on δP:

$$||\delta P|| \le 2(cond(T))^2||\delta A||||P_0||\int_0^\infty e^{2\mathfrak{Re}(\lambda_{max}(A_0))t}dt. \quad (7)$$

Here T is the matrix, which diagonalizes A. Since A is stable, the integral exists and finally we get an upper bound on infinitesimal perturbations of the gramian:

$$||\delta P|| \le \frac{1}{|\mathfrak{Re}(\lambda_{max}(A_0))|}(cond(T))^2||P_0||||\delta A|| \quad (8)$$

The norm of δP increases as the maximal eigenvalue of A_0 approaches imaginary axis. We evidently see that small perturbations in A will result in small perturbations in gramians.

4.2 Effect of perturbation of gramians on Cholesky factors

In our implementation we used an eigenvalue decompositions of P and Q in order to obtain Cholesky factorizations Z_c and Z_b.

$$P = UDU^T \tag{9}$$

Here D is a diagonal matrix. And we write Cholesky factorization to be

$$Z_B = UD^{1/2}U^T = (Z_B)^T; \tag{10}$$

This means that we actually compute a square root of P, which is symmetric and positive semidefinite. Let's represent $P = P_0 + \delta P$, $Z_B = Z_B^0 + \delta Z_B$ where Z_B^0, Z_B, P_0 and P are symmetric positive semidefinite. Neglecting second-order terms, we get:

$$\delta Z_B Z_B^0 + Z_B^0 \delta Z_B - \delta P = 0; \tag{11}$$

Assuming the unperturbed system to be controllable, which implies a positive definite Z_B, this equation has a unique solution. Moreover, similarly to the previous section, the 2-norm of the infinitesimal perturbation δZ_B will be:

$$||\delta Z_B|| \leq \frac{1}{2|\mathfrak{Re}(\lambda_{min}(Z_B^0))|}||\delta P|| \tag{12}$$

As one may expect, the closer matrix Z_B^0 to singular, the bigger the expected perturbation of Cholesky factors. Again, small δP will result in small δZ_B.

4.3 Balancing transformation

Consider that U_L satisfies a symmetric eigenvalue problem:

$$U_L \Sigma^2 U_L^T = Z_C^T P Z_C; \tag{13}$$

Below we summarize a perturbation theory for a symmetric eigenvalue problem with nondegenerate spectrum.

Consider a symmetric matrix $M = M_0 + \delta M$, where M_0 is unperturbed matrix with known eigenvalues and eigenvectors, and no repeated eigenvalues. Eigenvectors of M can be represented as a linear combinations of eigenvectors of M_0:

$$x_k = \sum_{i=1}^{N} c_i^k x_i^0, \tag{14}$$

where x_k is a k-th eigenvector of perturbed matrix M and x_i^0 is an i-th eigenvector of the unperturbed matrix. Coefficients c_i^k show how eigenvectors of matrix M_0 are intermixed due to perturbation δM. We have:

$$(M_0 + \delta M)\sum_{i=1}^{N} c_i^k x_i^0 = \lambda_k \sum_{i=1}^{N} c_i^k x_i^0 \Rightarrow$$
$$\sum_{i=1}^{N} c_i^k \delta M_{ji} = (\lambda_k - \lambda_j^0)c_j^k, \tag{15}$$

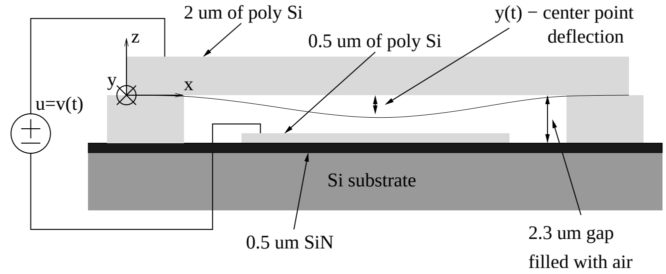

Figure 1: Micromachined switch (following Hung et al. [4]).

where λ_k and λ_k^0 are k-th eigenvalues of M and M_0 consequently and $\delta M_{ij} = (x_i^0)^T \delta M x_j^0$ is a matrix element of the perturbation in basis of unperturbed eigenvectors.

Now we assume small perturbations and represent $\lambda_k = \lambda_k^0 + \lambda_k^{(1)} + \lambda_k^{(2)} + ...$ and $c_k^n = \delta_{kn} + c_k^{n(1)} + c_k^{n(2)} ...$ where each subsequent term represents smaller orders in magnitude. For the first-order terms we get:

$$\lambda_k^{(1)} - \lambda_k^0 = \delta M_{jj} \tag{16}$$

and

$$c_k^n = \frac{\delta M_{kn}}{\lambda_n^0 - \lambda_k^0}, k \neq n; \tag{17}$$

The greater the separation between eigenmodes, the less they tend to intermix due to small perturbations. If some modes have eigenvalues which are close, they change rapidly with perturbation. Keeping this in mind, we propose the following recipe for choosing an order of projection basis:

Recipe for using TBR as a linear reduction for TPWL framework: We need to pick a reduced order to ensure that the remaining Hankel singular values are small enough and the last kept and first removed Hankel singular values are well separated.

5 COMPUTATIONAL RESULTS

In order to illustrate the importance of the described approach, we consider a micromachined switch device (fixed-fixed beam) shown in Figure 1. Following Hung et al. [6], the dynamical behavior of this coupled electro-mechanical-fluid system can be modeled with 1D Euler's beam equation and 2D Reynolds' squeeze film damping equation [6]. Spatial discretization of those equations using a standard finite-difference scheme leads to a nonlinear dynamical system in form (1) with $N = 880$ states. For the considered example we select our output $y(t)$ as the deflection of the center of the beam from the equilibrium point (cf. Figure 1).

We applied a TBR-based trajectory piecewise linear MOR procedure, using a single TBR reduction at the system initial state. Figure 2 shows the error in the output signal $\|y_r - y\|_2$, where y_r is the output signal computed with TBR-based TPWL reduced order model,

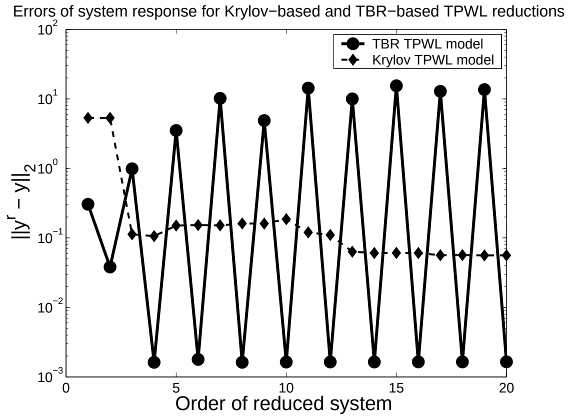

Figure 2: Errors in output computed by TPWL models generated with different MOR procedures; $N = 880$; 5.5-volt step testing and training input voltage.

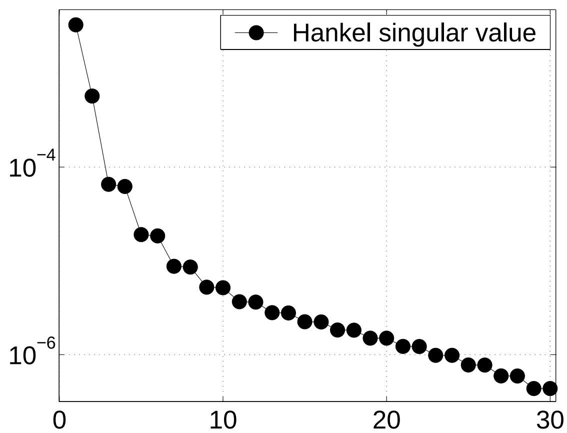

Figure 3: Hankel singular values of the balancing transformation at the initial state.

and y is computed with full order nonlinear model, for different orders q of the reduced model. The dashed line corresponds to a different choice of projection basis, namely the Krylov-subspace linear reduction.

Evidently, macromodels with odd orders behave very differently than the macromodels with even orders. Models of even order are substantially more accurate than models of odd order, the latter being unstable and inaccurate. This effect is perfectly consistent with our perturbation results if we take a look at figure 3, where the Hankel singular values of the balancing transfromation are plotted. The Hankel singular values for our example are arranged in pairs of values, and evidently, even-order models violate our recipe for choice of reduction basis.

6 CONCLUSIONS

In this paper we have analyzed the choice of order for TBR linear reduction for trajectory piecewise-linear nonlinear model order reduction. We showed that Hankel singular values of balancing transformation determine perturbation properties of the obtained projection basis. We suggested a simple recipe which helps to ensure that all linearizations are stably reduced.

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