# Solution of Large-Scale Networks by Tearing

FELIX F. WU, MEMBER, IEEE

*Abstract*—**A generalized method of tearing, or diakoptics, for solving large-scale networks is derived. The idea of diakoptics is viewed as simply the partition of branches and the Kirchhoff laws. A solution algorithm based on LU decomposition is presented. If the network may be "torn apart," then the computations at the subnetwork level can be carried out independently. We present an example which is a family of networks with a parameter $p$ and show that diakoptics, compared to the conventional network analysis, sometimes saves computations and sometimes requires more computations.**

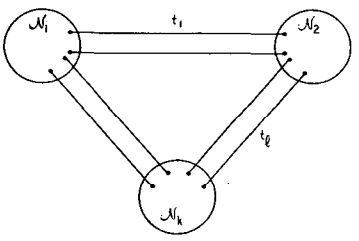


Fig. 1. Networks from which idea of tearing originated.

## I. Introduction

CONSIDER a network $\mathcal{N}$ consisting of many subnetworks $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_k$ interconnected together by branches $t_1, t_2, \cdots, t_l$ (Fig. 1). Such networks are common in practice, e.g., large-scale interconnected power systems. We may view the set of branches $t_1, \cdots, t_l$ as having the property that their removal tears the network apart into several independent subnetworks. The original suggestion of the method of tearing, or "diakoptics," is to solve the network problem in two steps: i) subnetwork level: one tears away the branches $t_1, t_2, \cdots, t_l$ and solves the subnetworks $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_k$ independently, ii) interconnection level: one combines these results with the branch variables associated with $t_1, t_2, \cdots, t_l$ to obtain the overall solution.

The idea of tearing was introduced by Kron [1]. He applied the concept to solve a certain class of networks. His derivation of the approach is based on the concepts from tensor analysis. Happ [2] has expanded the theory and applications along the same line. Kron's derivation is obscure. Branin [3], and Sasson and Brown [17] have attempted to clarify the concepts. Recently Chua and Chen [4] have shown that diakoptics can be derived from the generalized hybrid analysis.

We present a simple derivation of diakoptics. We view the basic idea of diakoptics as merely the partition of the branches and the Kirchhoff laws. Our version of diakoptics is more general and includes all the previous results as special cases.

Diakoptics was developed as an approach to solve large-scale networks. In practice large-scale networks are usually sparsely connected. We present a solution algorithm for diakoptic analysis which is based on LU decomposition and is suitable to incorporate sparse matrix techniques [5], [6]. The solution algorithm can be applied provided a generic condition on the network is satisfied. If the network may be "torn apart," then there are several steps in the algorithm that can be carried out independently.

It has been questioned whether the diakoptic approach saves computation when the sparse matrix techniques are employed. We use an example of a family of networks with a parameter $p$ to demonstrate that, compared to the conventional node analysis, diakoptics sometimes saves computations and sometimes requires more computations.

## II. Diakoptic Node Analysis

Let $\mathcal{N}$ be a connected network having $(n+1)$ nodes, $v=\{n_0, n_1, \cdots, n_n\}$, and $b$ branches, $\beta=\{b_1, b_2, \cdots, b_b\}$, with linear time-invariant elements and sinusoidal sources. Consider the network $\mathcal{N}$ in the sinusoidal steady-state.[1] Phasor notations will be used throughout this paper. Let the branch voltages and currents be denoted by $v=(v_1, v_2, \cdots, v_b)$ and $i=(i_1, i_2, \cdots, i_b)$, respectively. In the node analysis [7] one node is selected as the datum node.[2] The $n$ node-to-datum voltages $V=(V_1, V_2, \cdots, V_n)$ are used as network variables. The basic idea of tearing is to distinguish certain branches, henceforth called *tearing branches*,[3] from the *remaining branches*. The set of branches $\beta$ is thus partitioned into two classes, $\beta_t$ and $\beta_r$. We use subscript $r$ and $t$ to denote quantities pertaining to

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The author is with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

[1]We assume that the sinusoidal steady-state response of $\mathcal{N}$ exists.
[2]Our terminologies and most of the notations agree with [7].
[3]Any subset of branches may be chosen as tearing branches in our derivation. Hoewever to achieve the computational advantages it is desired to choose those branches whose removal will tear the network apart.

the remaining branches and the tearing branches, respectively, e.g., $v=(v_r,v_t)$ and $i=(i_r,i_t)$.

Each branch $k$ is assumed to have the general form [7, pp. 409–414], which may include a voltage source $v_{sk}$ and a current source $i_{sk}$, and mutual coupling may exist between branches. We shall however make the following assumption, which is desirable from computational viewpoint [8, Remark 2], though it is not necessary for the derivation.

*Assumption 1*

There is no mutual coupling between the tearing branches and the remaining branches.

Let $y_r$ denote the branch admittance matrix of the remaining branches and $z_t$ denote the branch impedance matrix of the tearing branches. The branch relations are expressed as

$$\begin{cases} i_r = i_{sr} + y_r v_r - y_r v_{sr} \\ v_t = v_{st} + z_t i_t - z_t i_{st}. \end{cases}$$

Let us also partition the set of nodes $v$. The removal of the tearing branches may result in many separate parts,[4] and one of them contains the original datum node. For each of the remaining separate parts, we pick a node as a reference node.[5] Let $v_c$ denote the set of all these reference nodes.[6] Let $v_o$ denote the set of all the other nodes that are not in $v_c$. Hence $v$ is partitioned into $v_o$ and $v_c$. We use subscript $o$ and $c$ to denote quantities pertaining to $v_o$ and $v_c$ respectively, hence $V=(V_o,V_c)$. The set of nodes that are connected only by the tearing branches is a subset of $v_c$.[7] For our later reference, we denote this set of nodes by $v_{c2}$ and denote the complement of $v_{c2}$ with respect to $v_c$ by $v_{c1}$.

The foregoing partition of branches and nodes gives rise to a natural partition of the reduced incidence matrix $A$[8]

$$\mathrm{A}= \begin{array}{c|cc} & \beta_r & \beta_t \\ \hline v_o & A_r & A_t \\ v_c & a_r & a_t \end{array}.$$

The network variables are constrained by the Kirchhoff current law (KCL), Kirchhoff voltage law (KVL), and the branch relations (BR). The constraints in node analysis are expressed as follows:

$$\text{(KCL)} \quad \begin{cases} A_r i_r + A_t i_t = 0 & (1) \\ a_r i_r + a_t i_t = 0 & (2) \end{cases}$$

$$\text{(KVL)} \quad \begin{cases} v_r = A_r^T V_o + a_r^T V_c & (3) \\ v_t = A_t^T V_o + a_t^T V_c & (4) \end{cases}$$

$$\text{(BR)} \quad \begin{cases} i_r = i_{sr} + y_r v_r - y_r v_{sr} & (5) \\ v_t = v_{st} + z_t i_t - z_t i_{st}. & (6) \end{cases}$$

We shall call the network resulted from $\mathfrak{N}$ by removing all the tearing branches and the nodes in $v_{c2}$, the *torn network* of $\mathfrak{N}$.

*Remark 1*

Suppose $v_{c2}$ is empty.[9] The KCL, KVL, and BR for the torn network are expressed as

$$\text{(KCL)} \quad \begin{cases} A_r i_r = 0 & (7) \\ a_r i_r = 0 & (8) \end{cases}$$

$$\text{(KVL)} \quad v_r = A_r^T V_o + a_r^T V_c \qquad (9)$$

$$\text{(BR)} \quad i_r = i_{sr} + y_r v_r - y_r v_{sr}. \qquad (10)$$

The standard procedure for node analysis in this case is to substitute (9) into (10), and then substitute the result into (7) and (8) to obtain equations relating the variables $V$.

The idea of tearing involves solving the torn network as the first step. Therefore, let us compare (7)–(10) of the torn network with (1)–(3), (5) and perform the same procedures of node analysis described above to (1)–(3), (5), i.e., we substitute (3) into (5) and then substitute the result into (1) and (2) to obtain (11) and (13) below. The remaining two equations (4) and (6) are combined into (12) below

$$A_r y_r A_r^T V_o + A_t i_t + A_r y_r a_r^T V_c = J_o \qquad (11)$$

$$A_t^T V_o - z_t i_t + a_t^T V_c = E \qquad (12)$$

$$a_r y_r A_r^T V_o + a_t i_t + a_r y_r a_r^T V_c = J_c \qquad (13)$$

or

$$\begin{bmatrix} A_r y_r A_r^T & A_t & A_r y_r a_r^T \\ A_t^T & -z_t & a_t^T \\ a_r y_r A_r^T & a_t & a_r y_r a_r^T \end{bmatrix} \begin{bmatrix} V_o \\ i_t \\ V_c \end{bmatrix} = \begin{bmatrix} J_o \\ E \\ J_c \end{bmatrix} \qquad (14)$$

where $J_o \triangleq A_r y_r v_{sr} - A_r i_{sr}$, $J_c \triangleq a_r y_r v_{sr} - a_r i_{sr}$, and $E \triangleq v_{st} - z_t i_{st}$. Equation (14) will be referred to as the *diakoptic*

[4]A maximal connected subnetwork of an unconnected network is called a *separate part* [7, p. 387].

[5]If a separate part contains more than one node, then the choice of the reference node can be arbitrary. However it may be desirable to pick one among the nodes which are connected by the tearing branches.

[6]As will be clear later, the node voltages associated with $v_c$ will be calculated at the interconnection level, and $v_o$ contains all the other nodes.

[7]An isolated node *is* a separate part.

[8]The rows of $A$ correspond to the nodes (deleting the datum) and the columns of $A$ correspond to the branches [7, p. 417].

[9]This assumption is only for notational convenience, otherwise $J_c$ in (13) would be $J_{c1}$, etc.

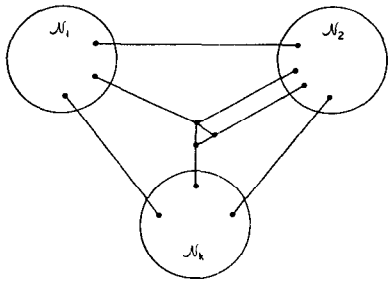

Fig. 2. Diakoptic analysis in this paper also applies to networks that have nodes connected only by tearing branches.

*node equation*. The process of arriving (14) and the solution of it will be referred to as the *diakoptic node analysis*.

*Remark 2*

Equation (14) can also be expressed as

$$\begin{bmatrix} A_r y_r A_r^T & A_t & A_r y_r a_r^T \\ A_t^T & -z_t & a_t^T \\ a_r y_r A_r^T & a_t & a_r y_r a_r^T \end{bmatrix} \begin{bmatrix} V_o \\ j_t \\ V_c \end{bmatrix} = \begin{bmatrix} J_o' \\ 0 \\ J_c' \end{bmatrix} \qquad (14')$$

where

$$j_t \triangleq i_t - i_{st} + z_t^{-1} v_{st}$$

$$J_o' \triangleq -A_r i_{sr} + A_r y_r v_{sr} - A_t i_{st} + A_t z_t^{-1} v_{st}$$

and

$$J_c' \triangleq -a_r i_{sr} + a_r y_r v_{sr} - a_t i_{st} + a_t z_t^{-1} v_{st}.$$

*Remark 3*

The coefficient matrix of (14) is easy to form. Note that the following four blocks

$$\begin{array}{|c|c|} \hline A_r y_r A_r^T & A_r y_r a_r^T \\ \hline a_r y_r A_r^T & a_r y_r a_r^T \\ \hline \end{array} \qquad (15)$$

are precisely the node admittance matrix of the torn network. It can be formed by the standard procedure.[10] The other five blocks are immediately obtained from the reduced incidence matrix and the branch relations.

*Remark 4*

Kron's [1] derivation of diakoptics is applicable to what he calls "diffusion-type" networks, or "radially-attached" networks, as Happ calls them. This is a class of networks for which the torn network is connected and hinged at the datum node. In terms of our approach, this class of networks corresponds to the special case where $v_c$ is empty. The diakoptic node equation then reduces to a simpler form [8]. Happ [9] has recently generalized Kron's derivation to include what he calls "radially unattached" networks. This class of networks corresponds to the special case where $v_{c2}$ is empty. Our general derivation of diakoptics in node analysis may include the case where there are nodes connected by tearing branches only (Fig. 2). This enlarges the applicability of diakoptics.

## III. Solution Algorithm

We will present a solution algorithm for the diakoptic node (14). Our solution algorithm is based on LU decomposition and is suitable to incorporate the sparse matrix techniques.

*Remark 5*

Sparse matrix equations are normally solved by the optimally ordered LU decomposition [10]. For a nonsingular matrix $A$, the existence of an LU decomposition is not always guaranteed unless we allow row and column permutations [10], [11, pp. 31–34]. It can be shown [15] that if sinusoidal steady state solutions exist for the three networks $\mathfrak{N}$, $\mathfrak{N}_a$, and $\mathfrak{N}_b$, then the coefficient matrix of (14), with the present block-ordering of rows and columns, is block LU-decomposable,[11] where $\mathfrak{N}_a$ is the network derived from $\mathfrak{N}$ by coalescing all the nodes of $v_c$ with the datum node (i.e., forcing the node voltages at $v_c$ to be zero) and $\mathfrak{N}_b$ is the network derived from $\mathfrak{N}_a$ by removing all the tearing branches.

In the derivation of the solution algorithm we shall make use of the following Fact, whose proof is in the Appendix.

*Fact 1*

$$a_r y_r a_r^T - a_r y_r A_r^T (A_r y_r A_r^T)^{-1} A_r y_r a_r^T = 0. \qquad (16)$$

To simplify notations, let $Y_{oo} \triangleq A_r y_r A_r^T$, $Y_{oc} \triangleq A_r y_r a_r^T$, and $Y_{co} \triangleq a_r y_r A_r^T$. Let us first LU-decompose $Y_{oo}$,

$$Y_{oo} = \mathrm{LU}. \qquad (17)$$

Equation (11) becomes

$$UV_o = L^{-1} J_o - L^{-1} A_t i_t - L^{-1} Y_{oc} V_c. \qquad (18)$$

Substituting $V_o$ from (18) into (12), we have

$$(z_t + A_t^T U^{-1} L^{-1} A_t) i_t = -E + A_t^T U^{-1} L^{-1} J_o + [a_t^T - A_t^T U^{-1} L^{-1} Y_{oc}] V_c. \qquad (19)$$

[10]For networks without coupling, see [7, p. 429]. Modification is needed to take care of the coupling.

[11]Note that LU decomposition does not exist if we interchange the second block rows and the third block of rows, and also the second block of columns and the third block of columns.

Let us define

$$\Phi_1 \triangleq L^{-1}A_t, \quad \Phi_2 \triangleq L^{-1}Y_{oc}, \quad \Psi_1 \triangleq A_t^T U^{-1},$$
$$\Psi_2 \triangleq Y_{co}U^{-1}, \quad \xi \triangleq L^{-1}J_o$$

and

$$F \triangleq z_t + \Psi_1\Phi_1. \tag{20}$$

Substituting (16), (18), and (19) into (13), we have

$$\left[(a_t - \Psi_2\Phi_1)F^{-1}(a_t^T - \Psi_1\Phi_2)\right]V_c = J_c - \left[\Psi_2\xi + (a_t - \psi_2\Phi_1)F^{-1}(\Psi_1\xi - E)\right]. \tag{21}$$

Therefore, we may solve $V_o$, $i_t$, $V_c$ in the following sequence:(17)-(20)-(21)-(19)-(18).

In the application of diakoptics, the case where the torn network has several "independent" subnetworks is of special interest. To be more specific, this is when the torn network is separable.[12] Let the torn network have $k$ separable subnetworks $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_k$. Let $\nu_o$ (respectively, $\beta_r$) be partitioned into $k$ classes $\nu_1, \nu_2, \cdots, \nu_k$ (respectively, $\beta_1, \beta_2, \cdots, \beta_k$), where $\nu_i$ (respectively, $\beta_i$) is the intersection of $\nu_o$ (respectively, $\beta_r$) and the nodes (respectively, branches) of $\mathcal{N}_i$. With this finer partition, we have $V_o = (V_1, V_2, \cdots, V_k)$, $J_o = (J_1, J_2, \cdots, J_k)$, and

$$A_r = \begin{array}{c} \\ \nu_1 \\ \nu_2 \\ \\ \nu_k \end{array} \begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & \cdots & \beta_k \end{array} \\ \begin{array}{|cccc} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{array} \end{array} \tag{22}$$

Suppose the following assumption holds.

*Assumption 3*

There is no mutual coupling between branches belonging to the different separable subnetworks.

Then the matrix $Y_{oo}$ takes a block diagonal form

$$Y_{oo} = \begin{array}{|ccc} Y_1 & & \\ & Y_2 & \\ & & \ddots \\ & & Y_k \end{array} \tag{23}$$

Let $A_t$, $A_t^T$, $Y_{oc}$, and $Y_{co}$ be partitioned into $(A_t)_i$, $(A_t^T)_i$, $(Y_{oc})_i$, and $(Y_{co})_i$, $i = 1, 2, \cdots, k$, accordingly (Fig. 3(a)). Note that the matrix is in a desired bordered block diagonal form (Fig. 3(b)) [6, p. 20].

[12]A network is *separable* if it has several separate parts and/or it is hinged [7, p. 445].

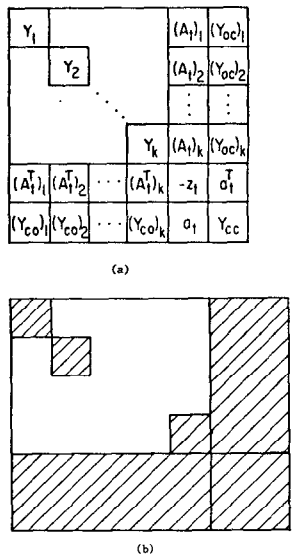


Fig. 3. (a) Partition of coefficient matrix according to $k$ separable subnetworks. (b) Bordered block diagonal form. Unshaded area consists of all zeros.

We may modify the previous solution procedure to take advantage of the decoupled block structure. Note that it is possible to perform computations of (17), (20), and (18) for each subnetwork independently. We arrive at the following solution algorithm.

*Solution Algorithm*

1) For each $i$, $i = 1, 2, \cdots, k$, do steps (S1) to (S6) as follows:

(S1) input $Y_i$, $(Y_{oc})_i$, $(Y_{co})_i$, $(A_t)_i$, $J_i$;
(S2) factor $Y_i = L_iU_i$;
(S3) solve $L_t\Phi_{1i} = (A_t)_i$ for $\Phi_{1i}$, $L_2\Phi_{2i} = (Y_{oc})_i$ for $\Phi_{2i}$; $\Psi_{1i}U_i = (A_t^T)_i$ for $\Psi_{1i}$, $\Psi_{2i}U_i = (Y_{co})_i$ for $\Psi_{2i}$;
(S4) solve $L_i\xi_i = J_i$ for $\xi_i$;
(S5) form $F_i = \Psi_{1i}\Phi_{1i}$, $G_i = \Psi_{1i}\Phi_{2i}$, $H_i = \Psi_{2i}\Phi_{1i}$; $g_i = \Psi_{1i}\xi_i$, $h_i = \Psi_{2i}\xi_i$;
(S6) output $F_i$, $G_i$, $H_i$, $g_i$, $h_i$.

2) Do steps (C1) to (C8) as follows:

(C1) input $z_t$, $a_t$, $J_c$, and $F_i$, $G_i$, $H_i$, $g_i$, $h_i$, $i = 1, 2, \cdots, k$;
(C2) form $F = z_t + \Sigma F_i$, $G = a_t^T - \Sigma G_i$, $H = a_t - \Sigma H_i$, $g = E + \Sigma g_i$, $h = \Sigma h_i$;
(C3) solve $KF = H$ for $K$;
(C4) form $P = KG$, $p = J_c - h - Kg$;
(C5) solve $PV_c = p$ for $V_c$;
(C6) form $f = g + GV_c$;
(C7) solve $Fi_t = f$ for $i_t$;
(C8) output $V_c$, $i_t$.

3) For each $i$, $i = 1, 2, \cdots, k$, do steps (S7) to (S10) as follows:

(S7) input $V_c$, $i_t$;
(S8) form $\zeta_i = \xi_i - \Phi_{1i}i_t - \Phi_{2i}V_c$;
(S9) solve $U_iV_i = \zeta_i$ for $V_i$;
(S10) output $V_i$.

*Remark 6*

The solutions of the triangular matrix equations in steps (S3), (S4), and (S9) are merely substitutions. Also note that since most of the columns of $(A_t)_i$ and $(Y_{oc})_i$, and the rows of $(Y_{co})_i$ are all zeros, computations for these zero rows and columns in Steps (S3) and (S5) need not be performed. Furthermore, in practical applications, the dimensions of $F$ and $P$ are usually small.

*Remark 7*

Parts 1) and 2) of the solution algorithm can be carried out independently for all the subnetworks. For example, a subroutine can be called repeatedly for $i=1, 2, \cdots, k$. Branin [16] has pointed out that the only significant computational advantage to be gained from piecewise method comes from economics made in the repeated use of the subnetwork LU factors.

*Remark 8*

Part 1) of the algorithm corresponds to the "subnetwork level" of diakoptics mentioned in Section I and parts 2) and 3) correspond to the "interconnection level" of diakoptics.

*Remark 9*

If $v_c$ is empty the algorithm reduces to a simpler version [8].

## IV. Computational Considerations

Applying the conventional node analysis to the network $\mathcal{N}$, we arrive at the following equation:

$$\left\{\begin{bmatrix} A_r & A_t \\ a_r & a_t \end{bmatrix}\begin{bmatrix} y_r & 0 \\ 0 & z_t^{-1} \end{bmatrix}\begin{bmatrix} A_r^T & a_r^T \\ A_t^T & a_t^T \end{bmatrix}\right\}\begin{bmatrix} V_o \\ V_c \end{bmatrix} = \begin{bmatrix} J_o' \\ J_c' \end{bmatrix}. \quad (24)$$

Where $J_o'$ and $J_c'$ are the equivalent current source vectors at $v_o$ and $v_c$ respectively, as defined in Remark 2. The coefficient matrix here is simply the node admittance matrix $Y$ of $\mathcal{N}$. Both the conventional node analysis (solution of (24)) and the diakoptic node analysis (solution of (14)) give us the node voltages. Naturally we would like to know which one requires less total computation to obtain the solution. For ease of later reference we will denote the matrix of (14) by $T$. In what follows, we count only multiplications in the comparison.[13]

For large-scale network, the matrices $Y$ and $T$ are very sparse. In sparse matrix computation, operations involving zero are not performed. Consider solving the sparse system $Ax=b$, where $A$ is $n\times n$ and $A=LU$, let $l_i$ denote the number of nonzero elements in the $i$th column of $L$, and $u_i$ denote the number of nonzero elements in the $i$th row of $u$. It can be shown [8], [12], by simple counting, that the total number of (complex) multiplications required to solve $Ax=b$ is equal to $\sum_{i=1}^{n}(l_i+1)u_i-2n$. Clearly the ordering of rows and columns of $A$ affects greatly the $l_i$'s and $u_i$'s. Several locally optimal ordering schemes [5], [6], are known.

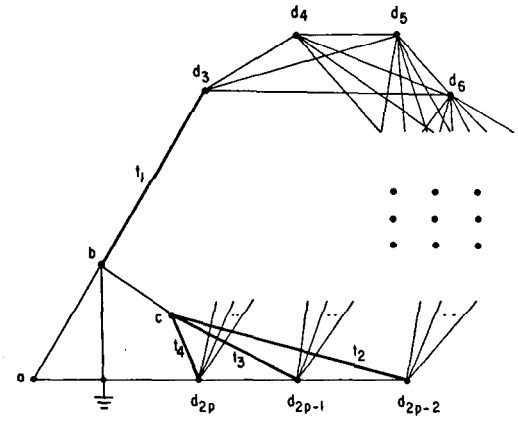


Fig. 4. Network for example. Branches in heavy lines are tearing branches.

We now present an example of a family of network having a parameter $p$. Depending on the value of $p$, the number of multiplications required for the diakoptic node analysis may be less than, or greater than that for the conventional node analysis, both with optimal ordering.[14] This clearly demonstrates that neither approach is absolutely superior to the other, as far as the total number of multiplications is concerned.

*Example*

Consider the network shown in Fig. 4. Let all the branches be two-terminal elements (no mutual coupling). The branches connecting nodes $d_i$'s are defined as follows:

i) for all $3 \leqslant i \leqslant p$, $i<j\leqslant 2i$, there is a branch connecting nodes $d_i$ and $d_j$;
ii) for all $p<i\leqslant 2p$, $i<j\leqslant 2p$, there is a branch connecting nodes $d_i$ and $d_j$.

For the conventional node analysis, we first form the matrix $Y$, which has the same sparsity structure as the network $\mathcal{N}$, i.e., the $ij$th element of $Y$, $U_{ij}$, is nonzero iff there is a branch in $\mathcal{N}$ connecting node $i$ and node $j$. It can be shown that by applying any of the locally optimal

[13]The formation of the matrices $Y$ and $T$ involves only additions. For node analysis, voltage sources have usually been transformed into current sources. Hence the formation of $J_o'$ and $J_c'$ involves only additions. We assume (14'), instead of (14), is used for diakoptic node analysis.

[14]We compare the computations required for the solution of the node voltages. Therefore the fact that the diakoptic approach gives, in addition to the node voltages, also the tearing-branch currents is not taken into account.

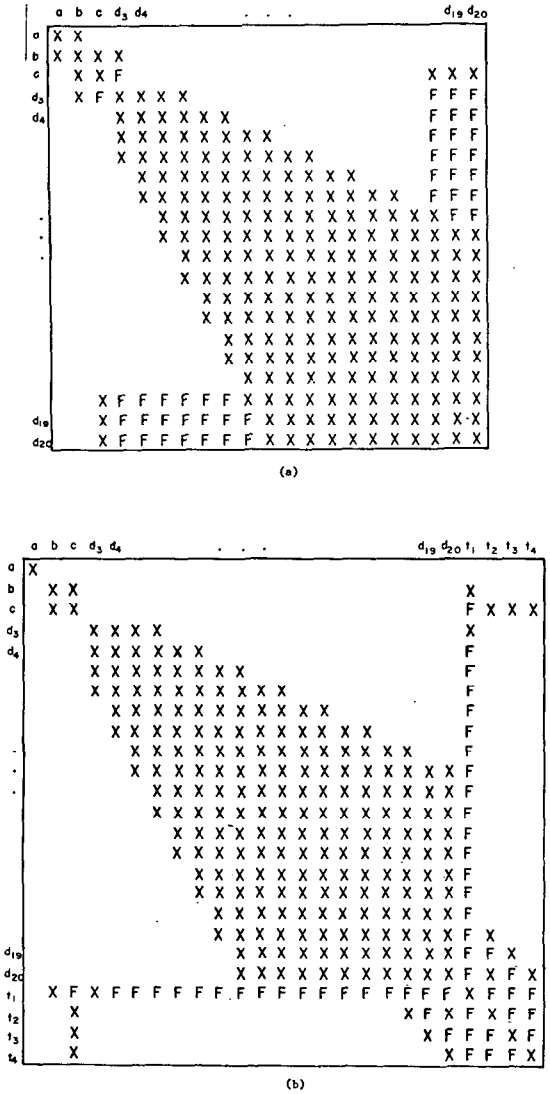

Fig. 5. (a) Nonzero pattern of $Y$ (nonzero elements of $Y$ are marked $X$) and fill-ins (marked $F$) introduced by the LU decomposition. (b) Nonzero pattern of $T$ and fill-ins introduced by LU decomposition.

ordering schemes [5] to $Y$, the ordering $(a, b, c, d_3, d_4, \cdots, d_{2p})$ will result. The nonzero pattern of $Y$, together with the fill-ins[15] is shown in Fig. 5(a). A little calculation shows that the total number of multiplications for the solution in this case is equal to $\gamma_p = 2\sum_{k=1}^{p+2} k(k+1) - 4p - 66$.

Now let us consider solving this problem by diakoptics. Suppose we pick the branches $(b,d_3)$, $(c,d_{2p-2})$, $(c,d_{2p-1})$, and $(c,d_{2p})$ as the tearing branches $t_1$, $t_2$, $t_3$, and $t_4$, respectively. Let us form the matrix $T$. Again it can be shown that within their blocks, $(a, b, c, d_3, \cdots, d_{2p})$ and $(t_1, t_2, t_3, t_4)$ are optimally ordered.[16] The nonzero pattern of $T$, together with the fill-ins, is shown in Fig. 5b. The total number of multiplications to obtain the solution in this case is equal to $\delta_p = 2\sum_{k=1}^{p+1} k(k+1) + (p+2)(p+3) - 4p + 88$. Hence $(\gamma_p - \delta_p) = p^2 + 5p - 148$.

If $p \geqslant 10$, $\gamma_p > \delta_p$, i.e., the diakoptic node analysis requires less multiplications. On the other hand, if $3 \leqslant p \leqslant 9$, $\gamma_p < \delta_p$, i.e., the diakoptic node analysis requires more multiplications.

## V. Other Diakoptic and Codiakoptic Analyses

There are several standard network analysis procedures, namely, node analysis, mesh analysis, cutset analysis, loop analysis, and mixed analysis. Each of them provides a systematic way of writing linearly independent Kirchhoff laws. Once it is understood that diakoptics involves merely partition of branches and the Kirchhoff laws, similar derivation as in Section II can be applied to other network analysis procedures. In the following, we will not repeat the obvious similarities. Only the diakoptic mixed analysis is derived in some detail. We call the dual of a diakoptic analysis *codiakoptic analysis* [13].

### *1) Diakoptic Cutset Analysis*

Given a tree of the network $\mathcal{N}$, the set of tree branches can be partitioned into two classes; i.e., those of the remaining branches and those of the tearing branches. The set of tree-branch voltages, which are the network variables in this case, is partitioned into $V_r$ and $V_t$, accordingly. Also the set of fundamental cutsets $\gamma$ is partitioned into $\gamma_r$ and $\gamma_t$. Hence the fundamental cutset matrix $Q$ takes the form

$$Q = \begin{array}{c|c|c} & \beta_r & \beta_t \\ \hline \gamma_r & Q_r & Q_t \\ \hline \gamma_t & q_r & q_t \end{array} . \tag{25}$$

Note that if we choose a tree such that it contains as many remaining branches as possible then $q_r = 0$. On the other hand, if we choose a tree such that it contains as many tearing branches as possible then $Q_t = 0$.

### *2) Codiakoptic Mesh Analysis*

This is the dual of the idakoptic node analysis for a planar network. Consider the set of meshes $\mu$. If the tearing branches form loops, we pick a mesh from the region enclosed by such a loop. Let $\mu_c$ denote the collection of such meshes. Let $\mu_o$ denote the set of other meshes. Hence $\mu$ is partitioned into $\mu_o$ and $\mu_c$ and the reduced mesh matrix $M$ takes the form

$$M = \begin{array}{c|c|c} & \beta_r & \beta_t \\ \hline \mu_o & M_r & M_t \\ \hline \mu_c & m_r & m_t \end{array} . \tag{26}$$

[15]The $ij$th position is said to be a *fill-in* if $Y = \mathrm{LU}$ and $Y_{ij} = 0$ but either the $ij$th element of $L$ (if $i > j$), or of $U$ (if $i < j$) is nonzero.

[16]The same letter, e.g., $d_i$, is used to denote the rows and columns corresponding to the node $d_i$ for both $Y$ and $T$.

The *dual torn network* is defined as the network resulted from $\mathcal{N}$ by contracting all the tearing branches.[17] In order to have the dual torn network having several "independent" subnetworks, the set of tearing branches should be so chosen that the dual torn network has hinged subnetworks and the subnetworks are not mutually coupled. For a planar network the set of tearing branches having this property can be characterized as

i) they divide the plane into several regions;
ii) there is no mutual coupling between branches belonging to different regions.

*3) Codiakoptic Loop Analysis*

This is the dual of the diakoptic cutset analysis. Given a tree of $\mathcal{N}$, the set of link currents (the network variables for loop analysis) is partitioned into $I_r$ and $I_t$. The fundamental loops $l$ are partitioned into $l_r$ and $l_t$. Hence

$$B = \begin{array}{c} \\ l_r \\ l_t \end{array} \begin{array}{c} \begin{array}{cc} \beta_r & \beta_t \end{array} \\ \begin{array}{|c|c|} \hline B_r & B_t \\ \hline b_r & b_t \\ \hline \end{array} \end{array}. \tag{27}$$

Note that if we choose a tree such that it contains as many remaining (respectively, tearing) branches as possible then $B_t = 0$ (respectively, $b_r = 0$).

*4) Diakoptic Mixed Analysis*

The essence of mixed analysis is that a set of independent Kirchhoff law equations is selected from two network analysis formulations. We will base our derivation below on the mixed cutset and loop analysis. We will comment on the generality of the approach later.

Let us first pick a tree of $\mathcal{N}$ such that it contains as many remaining branches as possible. Now we write down Kirchhoff laws for the cutset analysis and the loop analysis side by side[18]

| | Cutset analysis | | Loop analysis | |
|---|---|---|---|---|
| (KCL) | $Q_r i_r + Q_t i_t = 0$ | (28) | $i_r = B_r^T I_r + b_r^T I_t$ | (29) |
| | $q_t i_t = 0$ | (30) | $i_t = b_t^T I_t$ | (31) |
| (KVL) | $v_r = Q_r^T V_r$ | (32) | $B_r v_r = 0$ | (33) |
| | $v_t = Q_t^T V_r + q_t^T V_t$ | (34) | $b_r v_r + b_t v_t = 0$ | (35) |
| (BR) | $i_r = i_{sr} + y_r v_r - y_r v_{sr}$ | (36) | $i_r = i_{sr} + y_r v_r - y_r v_{sr}$ | (36) |
| | $v_t = v_{st} + z_t i_t - z_t i_{st}$ | (37) | $v_t = v_{st} + z_t i_t - z_t i_{st}$ | (37). |

Let $\mathcal{N}_r$ denote a network derived from $\mathcal{N}$ by removing all the tearing branches and $\mathcal{N}_t$ denote a network derived from $\mathcal{N}$ by contracting all the remaining branches. Note that both (32) and (33) are complete characterization of KVL constraints for the network $\mathcal{N}_r$, similarly, both (30) and (31) are complete characterization of KCL constraints for the network $\mathcal{N}_t$ [14]. Therefore, (28) and (31) form a complete set of KCL for $\mathcal{N}$ and (32) and (35) form a complete set of KVL for $\mathcal{N}$. We thus proceed our analysis for $\mathcal{N}$ with KCL constraints (28)–(31), KVL constraints (32)–(35), and BR (36), (37).

Motivated by the cutset analysis on $\mathcal{N}_r$ and the loop analysis on $\mathcal{N}_t$, we combine (28), (32), (36) and (31), (35), (37). We then eliminate $i_t$ and $v_r$ from the expressions by substituting (31) and (32). Thus we obtain

$$\begin{bmatrix} Q_r y_r Q_r^T & Q_t b_t^T \\ b_r Q_r^T & b_t z_t b_t^T \end{bmatrix} \begin{bmatrix} V_r \\ I_t \end{bmatrix} = \begin{bmatrix} J_r \\ E_t \end{bmatrix} \tag{38}$$

where $J_r \triangleq Q_r y_r v_{sr} - Q_r i_{sr}$ and $E_t \triangleq b_t z_t i_{st} - b_t v_{st}$. Note that $b_r Q_r^T = -b_t Q_t^T$ as a consequence of the fact that $BQ^T = 0$ [7, p. 493]. We may interpret $J_r$ and $E_t$ as follows. Suppose we first transform the voltage (current) source associated with a remaining (tearing) branch into an equivalent current (voltage) source. Then $J_r(E_t)$ is the sum of all equivalent current (voltage) source in the fundamental cutsets (loops) defined by the remaining-branch-tree-branches (tearing-branch links).

*Remark 10*

Note that the cutset analysis (28) merely provides a set of linearly independent KCL for the network $\mathcal{N}_t$ and (32) provides a complete characterization of the KVL constraints for $\mathcal{N}_t$. We certainly may replace (28) by a set of linearly independent KCL for $\mathcal{N}_t$ supplied by the node analysis equations and also replace (32) by the corresponding node voltage characterization of KVL constraints from node analysis.[19] Similarly for planar networks, the loop analysis (31) and (35) may be replaced by the mesh analysis equations. Thus we may have diakoptic mixed node-and-loop (or node-and-mesh) analysis.

[17] A branch is *contracted* if its two end nodes are coalesced into one node and the branch itself is then removed.

[18] Different trees may be chosen for the cutset analysis and the loop analysis so long as each tree contains maximal number of remaining branches.

[19] If there are several separate parts in $\mathcal{N}_t$, node analysis should be performed for each separate part.

*5) Codiakoptic Mixed Analysis*

This is the dual of the diakoptic mixed analysis. Here we start by picking a tree of $\mathcal{N}$ such that it contains as many tearing branches as possible.

## APPENDIX

*Proof of Fact 1*

We first prove the fact that each row of $a_r$ is a linear combination of the rows of $A_r$, i.e., $a_r = CA_r$.

Consider a row in $a_r$, which corresponds to a node in $v_{c1}$, i.e., this row is associated with the node which is picked from a separate part $\mathcal{N}_i$ of the torn network. This row is indeed the negative of the sum of those rows in $A_r$ that correspond to the nodes in $\mathcal{N}_i$, because all these rows together form the (unreduced) incidence matrix of $\mathcal{N}_i$. Next consider a row in $a_r$ that corresponds to a node in $v_{c2}$. It is a zero row, which is a trivial linear combination of the rows of $A_r$. Hence, $a_r = CA_r$.

Now we substitute $a_r = CA_r$ into the left-hand side of the expression (16) and obtain

$$a_r y_r a_r^T - a_r y_r A_r^T \left(A_r y_r A_r^T\right)^{-1} A_r y_r a_r^T$$
$$= CA_r y_r A_r^T C^T - CA_r y_r A_r^T \left(A_r y_r A_r^T\right)^{-1} A_r y_r A_r^T C^T = 0.$$

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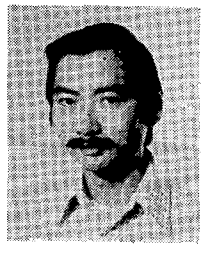


**Felix F. Wu** (S'69–M'73) received the B.S. degree from the National Taiwan University, Taipei, Taiwan, in 1965; the M.S. degree from the University of Pittsburgh, Pittsburgh, PA, in 1969, both in electrical engineering; and the Ph.D degree in electrical engineering and computer sciences from the University of California, Berkeley, in 1972.

From 1972 to 1974, he was an Assistant Professor with the Department of Electrical Engineering, University of Pittsburgh. Since July 1974, he has been an Assistant Professor with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley. His current research interest is in power system analysis.

Dr. Wu is member of the Computer and Analytical Methods Subcommittee, Power System Engineering Committee, and the Education Resources Subcommittee, Power Engineering Education Committee, of the IEEE Power Engineering Society. He is also a member of Eta Kappa Nu, Tau Beta Pi, Sigma Xi, and the American Association for the Advancement of Science.