A Two Levels Algorithm for Tearing

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Abstract—**This paper deals with tearing methods for the solution of a large scale system of linear algebraic equations. A modification algorithm is presented and evaluated with respect to other available techniques, namely, Householder's formula and Bennet's algorithm. Then, an optimization problem related to the "best" way of tearing a given matrix A with a certain associated structure is stated and solved by proving it to be equivalent to the determination of a minimum essential set of a suitably defined hypergraph H. A branch-and-bound algorithm for minimum essential set in H, based on a number of local reduction rules is outlined. Finally, the application of the obtained results to the tearing problem is discussed and its complexity compared with LU decomposition method.**

I. Introduction

Recently much effort has been devoted to tearing methods for the analysis of large scale electrical networks [1]–[5].

Tearing, usually referred to as diakoptics, was introduced by Kron [6] and basically consists in breaking the original analysis problem into simple subproblems which can be solved either independently or according to a (possibly partial) ordering.

The basic problem dealt with in this paper is simply finding $x \in R^n$ such that

$$Ax = b \tag{1.1}$$

where $A \in R^{n^2}$ and $b \in R^n$ are given.

As is well known, the classical tearing technique basically consists of two steps. First, a $(n \times n)$-matrix C has to be found such that

i) the nonzero entries of C are equal to the corresponding ones in A;

ii) $B \triangleq A - C$ is nonsingular and generally such as to make the solution of $Bx = b$ a relatively simple task.

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Then, the solution of $Bx = b$ is modified to obtain the solution of system (1.1) by taking into account the "perturbation" due to the nonzero entries of the "cut matrix" C.

Such an approach is in fact followed in [5] by making use of Householder's formula. However, solving the problem via matrix inversion is, in general, a costly technique which furthermore fully destroys, with the original sparsity of A, any possibility of saving computer time and storage. In [6] these difficulties have been encompassed by a method related to Bennet's algorithm [7]. This algorithm consists in computing the LU factorization of A in terms of C and the LU factorization of B.

In this connection, the following terminology is here adopted. If a problem $P(\cdot)$ depends on a $(n \times n)$-matrix $A = B + C$, B and C being given, any method leading to $P(A)$ by adding to $P(B)$ the appropriate correcting term is generally called a modification method [5]–[10].

An extension to the multiple case of the modification algorithm described in [10] is given in Section II and a few comments about its computational complexity, as compared with other existing modification algorithms, are presented in Section III. Section IV deals with the problem of optimal tearing, i.e., with the problem of determining a cut matrix C such that a suitable measure of the overall computational effort involved by the method above is minimized. Under reasonable assumptions, this problem is shown to be equivalent to the one of finding a minimum essential set (MES) in a hypergraph H associated to A. In Section V, a particular way of applying the results obtained in Section II to a matrix reordered according to the criterion given in Section IV is described together with some computational remarks.

II. A Two Levels Modification Algorithm

Consider a system of linear algebraic equations of the form

$$(B + C)x = b \tag{2.1}$$

where $B \in R^{n^2}$ is nonsingular, $C \in R^{n^2}$ has rank m and is such that $A \triangleq B + C$ is nonsingular. Furthermore, let $C = HK'$, where $H, K \in R^{nm}$, and denote by I_k the identity matrix of order k. Then, the following theorem can easily be proven by extending to the multiple case $(m > 1)$ a similar result given in [10].

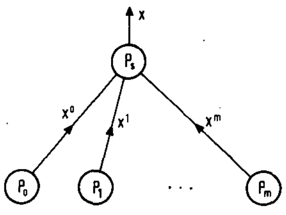

Fig. 1. Two levels structure of MA.

Theorem 2.1

Let $x^0 \in R^n$, $X \in R^{nm}$, and $w \in R^m$ be such that

$$B|x^0 \mid X| = |b \mid H| \tag{2.2}$$

$$(I_m + K'X)w = K'x^0 \tag{2.3}$$

then, the solution of system (2.1) is given by

$$x = x^0 - Xw.$$

Proof: By definition of x^0, X, and w, it follows that

$$x^0 = B^{-1}b$$
$$X = B^{-1}H$$
$$w = (I_m + K'X)^{-1}K'x^0 = \left(I_m + K'B^{-1}H\right)^{-1}K'B^{-1}b.$$

In fact [9], the nonsingularity of A and B and m being the rank of C imply the nonsingularity of $(I_m + K'B^{-1}H)$, while Householder's formula yields

$$A^{-1} = B^{-1} - B^{-1}H\left(I_m + K'B^{-1}H\right)^{-1}K'B^{-1}$$

hence

$$x = A^{-1}b = x^0 - Xw$$

and the theorem is proven. □

Theorem 2.1 induces in a straightforward way the following Modification Algorithm (MA).

Let $h^0 \triangleq b$, x^j and h^j be the jth column of H and X, respectively. Furthermore, set $i = 0$.

Step 1: Compute the LU factorization of B.

Step 2: Compute (by forward elimination and back substitution) x^i such that $Bx^i = h^i$.

Step 3: If $i = m$, go to Step 4; otherwise $i = i + 1$ and go to Step 2.

Step 4: Compute the LU factorization of $I_m + K'X$.

Step 5: Compute (by forward elimination and back substitution) w such that $(I_m + K'X)w = K'x^0$.

Step 6: Compute $x = x^0 - Xw$.

Remark 2.1: From a conceptual point of view, MA is a two levels algorithm the structure of which is shown in Fig. 1, where the subproblems P_s and P_i, $i = 0, 1, \cdots, m$, are defined as follows:

$$P_i : Bx^i = h^i, \qquad i = 0, 1, \cdots, m$$

$$P_s : (I_m + K'X)w = K'x^0, \qquad x = x^0 - Xw. \qquad \square$$

Remark 2.2: The relationships between Theorem 2.1 and the general theory of large scale systems are stronger and deeper than Remark 2.1 seems to show. As a matter of fact, the use of Householder's formula in the proof of Theorem 2.1 is just an unnecessary shortcut; deeper insight into the logical frame of the method can actually be gained by looking at the solution of the subproblems P_i, $i = 0, 1, \cdots, m$, as to an identification of the relationship between the "prediction" of an interaction vector and its "correspondingly resulting" value [11]–[13]. More specifically if system (2.1) is written as $Bx = b - HK'x = b - Hv$ where $v \triangleq K'x$, then it is straightforward to see that, taking v as an interaction vector, the original problem is equivalent to finding $\hat{v} \in R^m$ such that $K'\hat{x}$ is equal to $\hat{v}$, where $B\hat{x} = b - H\hat{v}$. To do that, the (linear) relationship between $\hat{v}$ and $K'\hat{x}$ must be preliminarily identified and this can be done by $m+1$ "experiments" consisting in taking first $\hat{v} = 0$ (see P_0, in Remark 2.1), then $b = 0$ and $\hat{v}$ equal to the ith orthonormal negative versor in R^m, $i = 1, 2, \cdots, m$ (see P_i, in Remark 2.1). Finally, imposing that $K'(x^0 + X\hat{v})$ be equal to $\hat{v}$, the "exact" value of $\hat{v}$ can be determined as well as the corresponding value of x (see P_s, in Remark 2.1). □

III. Computational Remarks

In this section, the complexity of MA is analyzed and subsequently compared with other available modification techniques, namely, Householder's formula and Bennet's algorithm.

Assumption 3.1: The number of multiplications required by a method is considered as its complexity measure. (Inversions are counted as multiplications.) □

Assumption 3.2: $n \gg m$. □

Under these assumptions, Step 1 of MA requires $n + \sum_{k=1}^{n-1} \zeta_k(\gamma_k + 1)$ operations where $\zeta_k + 1$ is the number of nonzero elements in the first row and $\gamma_k + 1$ is the number of nonzero elements in the first column of the reduced matrix of order $n - k + 1$ during the kth step of Gaussian elimination performed in natural order on B [10]. Step 2 requires $\beta = n + \sum_{k=1}^{n-1}(\gamma_k + \zeta_k)$ operations; since it has to be repeated $m+1$ times (Step 3), $(m+1)$ operations are globally required. Step 4 requires m^2n operations to compute $K'X$. The LU factorization of $I_m + K'X$ (which is generally full) and Step 5 require globally $1/3m^3 + m^2 - 1/3m$ operations; Step 6 requires mn operations.

In Table I, the above results are summarized and compared with the corresponding ones relative to the direct application of Householder's formula and Bennet's algorithm, under two different assumptions (full or sparse) concerning B.

It has to be noted that in the case of full matrices MA requires about half the time required by Bennett's algorithm.

TABLE I
COMPARISON OF HOUSEHOLDER'S FORMULA, BENNET'S ALGORITHM, AND MA

	Initial inversion or decomposition		Initial solution		Solution of the modified system	
	full	sparse	full	sparse	full	sparse
Householder	n^3	n^3	n^2	n^2	$3mn^2$	$3mn^2$
Bennett	$1/3\ n^3$	$\sum_{k=1}^{n-1} (\gamma_k\zeta_k+\zeta_k)$	n^2	$\sum_{k=1}^{n-1} (\gamma_k+\zeta_k)+n$	$(2m+1)n^2$	$3m^2n+(2m+1)\sum_{k=1}^{n-1}(\gamma_k+\zeta_k)$
MA	$1/3\ n^3$	$\sum_{k=1}^{n-1} (\gamma_k\zeta_k+\zeta_k)$	n^2	$\sum_{k=1}^{n-1} (\gamma_k+\zeta_k)+n$	mn^2	$m^2n + m\sum_{k=1}^{n-1}(\gamma_k+\zeta_k)$

IV. OPTIMAL DECOMPOSITION PROBLEM

This section deals with the important case where the system under consideration consists of a number of interconnected subsystems; i.e., the case where the unknown vector x can be "*a priori*" thought as "naturally" partitioned into a number of subvectors with an interaction pattern which is very strong among the elements of the same subvector and relatively weak among elements of different subvectors.

In this situation, it is generally conceivable to look for a "cut matrix" C such that $B \triangleq A - C$ is reducible to a block lower triangular form; consequently, the solution of $Bx=b$ (see Step 1 and Step 2 of MA) does actually decompose into a number of simpler and partially ordered subproblems corresponding to the strongly connected components of the digraph associated to B [10]. Such a decomposition can actually be done in more than a single way, so that an obviously important task is to find, among all the possible decompositions, at least one of those which are optimal in some specific sense. Problems of this kind are, in general, extremely difficult to solve and it is even hard to find situations where the computational effort needed to solve it is worth to be paid. However, it is quite reasonable, when searching for an optimal decomposition, to restrain the attention to some suitable subclass of all possible decompositions, thus resulting in a computationally feasible and economically efficient procedure. The herein adopted approach consists in restraining the search for an optimal decomposition only to those decompositions which retain, in a sense, the natural structure of the given system, which is supposed to originally consist of a number of well identified interacting subsystems.

In order to specifically state the optimization problem and illustrate the corresponding decomposition procedure, it is necessary to introduce some further definitions and notations.

Let Q_k be the set of the first k integers, Σ_k be the class of all ordered sets of k elements, $\mathcal{P}_k$ and Π_{kh} be the classes of all possible permutations from Σ_k to Σ_k and partitions from Σ_k to Σ_h, $h \leq k$, respectively.

Definition 4.1: A partition $\pi(\cdot)\in\Pi_{kh}$ is said to be regular if

$$\pi(Q_k)=\{\{1,2,\cdots,j_1\},\{j_1+1,j_1+2,\cdots,j_2\},\cdots,\{j_{h-1}+1,j_{h-1}+2,\cdots,k\}\}$$

for some $j_1,j_2,\cdots,j_{h-1} \in Q_k$, $0 \triangleq j_0 < j_1 < j_2 < \cdots < j_{h-1} < j_h \triangleq k$. □

Remark 4.1: Any regular partition $\bar{\pi}(\cdot)\in\Pi_{nq}$ induces in an obvious way a corresponding partition $\tilde{\pi}(\cdot)$ on R^{n^2}; formally

$$\tilde{\pi}(\cdot): A \mapsto \tilde{A}$$

$$\tilde{a}_{rs} = \begin{vmatrix} a_{j_{r-1}+1,j_{s-1}+1} & \cdots & a_{j_{r-1}+1,j_s} \\ \vdots & & \\ a_{j_r,j_{s-1}+1} & \cdots & a_{j_r,j_s} \end{vmatrix}, \qquad \forall r,s\in Q_q. \quad □$$

Definition 4.2: A matrix $A\in R^{n2}$ is block lower triangular with respect to a regular partition $\bar{\pi}(\cdot)\in\Pi_{nq}$ if $\tilde{A}$ is lower triangular; i.e., if $\tilde{a}_{rs}=0$ whenever $r<s$; $r,s\in Q_q$. □

Proposition 4.1: For any $\pi(\cdot)\in\Pi_{kh}$ there exist a unique regular partition $\bar{\pi}(\cdot)\in\Pi_{kh}$ and a unique permutation $p(\cdot)\in\mathcal{P}_k$ such that $\pi(\cdot)=\bar{\pi}(p(\cdot))$. □

Remark 4.2: Proposition 4.1 says that any partition is a regular partition of a permutation. The partition $\bar{\pi}(\cdot)$ and the permutation $p(\cdot)$ will henceforth be referred to as the regular partition and the permutation defined by $\pi(\cdot)$. □

Remark 4.3: Any permutation $p(\cdot)\in\mathcal{P}_n$ induces in an obvious way a corresponding (symmetric) permutation $\hat{p}(\cdot)$ on R^{n^2}; formally

$$\hat{p}(\cdot): A \mapsto EAE'$$

where $E\in R^{n^2}$ is a unimodular matrix defined as follows. Let $\{p_1,p_2,\cdots,p_n\}\triangleq p(Q_n)$, then $e_{ij}=\delta_{p_i,j}$, $\forall i,j\in Q_n$, where $\delta_{r,s}$ is the Kronecker function. □

Definition 4.3: For any $\pi(\cdot)\in\Pi_{nq}$, let $T_\pi(\cdot)\triangleq\tilde{\pi}(\hat{p}(\cdot))$ where $\tilde{\pi}(\cdot)$ and $\hat{p}(\cdot)$ are induced by the regular partition $\bar{\pi}(\cdot)$ and the permutation $p(\cdot)$ defined by $\pi(\cdot)$. □

Definition 4.4: A matrix $A\in R^{n^2}$ is block reducible relative to $\pi(\cdot)\in\Pi_{nq}$ if there exists $p^0(\cdot)\in\mathcal{P}_q$ such that $T_{p^0(\pi)}(A)$ is lower triangular. □

Remark 4.4: Assume that system (1.1) consists of q interconnected subsystems and that this kind of structural information can be specified by means of a regular partition $\bar{\pi}(\cdot)\in\Pi_{nq}$. If A is block reducible relative to $\bar{\pi}(\cdot)$, then splitting the problem into an equivalent set of q (partially ordered) subproblems is an almost trivial task. If this is not the case, a conceivable approach, in view of Theorem 2.1 consists in looking for a minimum rank matrix C such that $A-C$ is block reducible relative to a partition $\pi(\cdot)\in\Pi_{n\mu}$, $\mu\leq n$, the largest element of which has a cardinality significantly less than n. This kind of approach is followed in the sequel, where an optimal decomposition problem is formally stated and solved. □

Given $A\in R^{n^2}$ and $S\subset Q_n$, let $\pi(\cdot)\in\Pi_{n2}$ be such that $\pi(Q_n)=\{Q_n-S,S\}$; furthermore, let $\tilde{\tilde{A}}\triangleq T_\pi(A)$; then, define $A_S\triangleq\tilde{a}_{11}$. In other words, A_s is the matrix obtained from A by removing all rows and columns whose index is in S.

Given $\pi(\cdot)=\{\pi_1(\cdot),\pi_2(\cdot),\cdots,\pi_q(\cdot)\}\in\Pi_{nq}$ and $S\in Q_n$, with $|S|=m$, let $\pi_S(\cdot)\in\Pi_{\gamma q}$, $\gamma\triangleq n-m$, be such that

$$\pi_S(Q_n-S)=\{\pi_1(Q_n)-S,\pi_2(Q_n)-S,\cdots,\pi_q(Q_n)-S\}.$$

Definition 4.5: For any $S\subset Q_n$, the matrix $A\in R^{n^2}$ is block S-reducible relative to $\pi(\cdot)\in\Pi_{nq}$ if there exists $p^*(\cdot)\in\mathcal{P}_q$ such that $T_{p^*(\pi_S)}(A_S)$ is lower triangular; namely, if A_s is block reducible relative to $\pi_s(\cdot)$. □

Proposition 4.2: Let $A\in R^{n^2}$, $S\subset Q_n$, and $\pi(\cdot)\in\Pi_{nq}$. If A is block S-reducible relative to $\pi(\cdot)$, then there exists $\pi^*(\cdot)\in\Pi_{n2}$ such that $\tilde{a}_{11}$ is block lower triangular with respect to the regular partition defined by $p^*(\pi_S(\cdot))$, where $\tilde{\tilde{A}}=T_{\pi^*}(A)$. □

An optimization problem which, in view of Theorem 4.1, Remark 4.4, and Proposition 4.2, is of obvious interest can now be formally stated as follows.

Optimal Decomposition Problem (ODP)

Given $A\in R^{n^2}$ and $\pi(\cdot)\in\Pi_{nq}$ find $S\subset Q_n$ of minimal cardinality such that A is block S-reducible relative to $\pi(\cdot)$. □

In order to solve the problem above, it is quite natural to restate it in graph-theoretical terms.

A *directed hypergraph* $H=(X,Y)$ is constituted by a *node* set X and an *arc* set Y the elements of which are ordered pairs of nonempty subsets of X. Let $\Omega=\{\omega_1,\omega_2,\cdots,\omega_t\}$, $t>1$, be an ordered subset of X. If, for each $\omega_i\in\Omega$, there exists a pair Z_-^i and Z_+^i of subsets of X such that

$$\omega_i\in Z_-^i\cap Z_+^i$$

$$\eta_i\triangleq(Z_+^i,Z_-^{i+1})\in Y,\qquad \eta_i\neq\eta_j,\ \forall i,j\in Q_t,\qquad i\neq j$$

where

$$Z_-^{t+1}\triangleq Z_-^i$$

then Ω is a *cycle* of H of length t and the η_i's are called arcs of the cycle. A directed hypergraph without cycles is said to be *acyclic*.

Given a directed hypergraph $H=(X,Y)$ and a subset Z of X, the *section hypergraph of H with respect to Z* is an hypergraph $H_Z=(Z,Y_Z)$, where

$$Y_Z\triangleq\{(\tilde{X}_i,\tilde{X}_j)|\tilde{X}_i=X_i\cap Z,\tilde{X}_j=X_j\cap Z,(X_i,X_j)\in Y\}.$$

Any subset S of X is an *essential set of H* if H_{X-s} is acyclic. An essential set of minimum cardinality is said to be a *minimum essential set*. Its cardinality is said to be the *index* of H.

Given a matrix $A\in R^{n^2}$ and a partition $\pi(\cdot)=\{\pi_1(\cdot),\pi_2(\cdot),\cdots,\pi_q(\cdot)\}\in\Pi_{nq}$, let $H(A,\pi)$ be the direct hypergraph relative to A and $\pi(\cdot)$ defined as follows: $H(A,\pi)=(X,Y)$ where $X=Q_n$, $Y=Y_{\text{ex}}\cup Y_{\text{in}}$ where

$$Y_{\text{ex}}=\{(r,s)|r\in\pi_j(Q_n),s\in\pi_i(Q_n),i,j\in Q_q,i\neq j,a_{rs}\neq 0\}$$

and

$$Y_{\text{in}}=\{y_{\text{in}}^j|y_{\text{in}}^j=(\pi_j(Q_n),\pi_j(Q_n))\ \forall j\in Q_q\}.$$

The arcs in Y_{ex} are called *external arcs*, the arcs in Y_{in} are called *internal arcs*. It has to be noted that $|Y_{\text{in}}|=q$.

Lemma 4.1

For any $A\in R^{n^2}$, $\pi(\cdot)\in\Pi_{nq}$ and $p(\cdot)\in\mathcal{P}_q$, $H(A,\pi)$ is isomorphic to $H(A,p(\pi))$. □

Lemma 4.2

For any $A\in R^{n^2}$, $\pi(\cdot)\in\Pi_{nq}$, let $\bar{\pi}(\cdot)$ and $p(\cdot)$ be the regular partition and the permutation defined by $\pi(\cdot)$; i.e., $\pi(\cdot)=\bar{\pi}(p(\cdot))$. Then $H(A,\pi)$ is isomorphic to $H(\hat{p}(A),\bar{\pi})$ where $\hat{p}(\cdot)$ is the (symmetric) permutation on R^{n^2} induced by $p(\cdot)$. □

Lemma 4.3

For any $A\in R^{n^2}$ and $\pi(\cdot)\in\Pi_{nq}$, $H(A,\pi)$ is acyclic if and only if A is block reducible relative to $\pi(\cdot)$.

Proof: If A is block reducible relative to $\pi(\cdot)$, then (see Definition 4.4) there exists $p^0(\cdot)\in\mathcal{P}_q$ such that $T_{p^0(\pi)}(A)$ is lower triangular. Thus if $\pi(\cdot)$ and $p(\cdot)$ are the regular partition and the permutation defined by $p^0(\pi(\cdot))$, and $\hat{p}(\cdot)$ is defined as in Remark 4.3 then the hypergraph $H(\hat{p}(A),\bar{\pi})$ is acyclic since none of its external arcs is going from any node of $\bar{\pi}_i(Q_n)$ to any node of $\bar{\pi}_j(Q_n)$, whatever i, $j\in Q_q$ may be, with $i>j$. Since, in view of Lemma 4.2, $H(\hat{p}(A),\bar{\pi})$ is isomorphic to $H(A,p^0(\pi))$ and, in view of Lemma 4.1, $H(A,p^0(\pi))$ is isomorphic to $H(A,\pi)$, the conclusion can be drawn that $H(A,\pi)$ is acyclic.

Conversely, if $H(A,\pi)$ is acyclic, then there exists $p^\infty(\cdot)\in\mathcal{P}_q$ such that, letting $\pi^\infty(\cdot)\triangleq p^\infty(\pi(\cdot))$, $H(A,\pi^\infty)$ has no external arcs going from any node of $\pi_i^\infty(Q_n)$ to any node of $\pi_j^\infty(Q_n)$, whatever $i,j \in Q_q$ may be, with $i>j$. This means that $T_{p^\infty(\pi)}(A)$ is lower triangular, hence A is block reducible relative to $\pi(\cdot)$. □

Theorem 4.1

For any $A \in R^{n^2}$, $S \subset Q_n$, and $\pi(\cdot) \in \Pi_{nq}$, A is block S-reducible relative to $\pi(\cdot)$ if and only if S is an essential set of $H(A,\pi)$.

Proof: The hypergraph $H_{X-S}(A,\pi)$ is acyclic if and only if S is an essential set of $H(A,\pi)$. On the other hand, $H_{X-S}(A,\pi) = H(A_S, \pi_S)$. Hence, in view of Lemma 4.3, S is an essential set of $H(A,\pi)$ if and only if A_S is block reducible relative to $\pi_S(\cdot)$, i.e. (see Definition 4.5) if and only if A is block S-reducible relative to $\pi(\cdot)$. □

Corollary 4.1: Given $A \in \boldsymbol{R}^{n^2}$ and $\pi(\cdot) \in \Pi_{nq}$, ODP is equivalent to the determination of a minimum essential set of $H(A,\pi)$. □

It has to be noted that the problem of finding a minimum essential set in $H(A,\pi)$ can be easily proven to be hard.[1] In fact, the same problem for digraphs (also referred to as the minimum feedback vertex set problem) is well known to be hard [14]. The minimum essential set problem has been investigated in a number of papers and some satisfactory approaches have been developed. In particular, preliminary simplifications [15]–[17], branch and bound techniques [15]–[18] and near optimal algorithms [17], [18] have been devised.

In the sequel, some definitions are introduced so as to extend to $H(A,\pi)$ some results obtained to solve the minimum feedback vertex set problem. This extension is possible due to the very particular structure of the hypergraph associated to A.

Definition 4.6: The elimination of $x \in \pi_j(Q_n)$ from $H(A,\pi)$ is accomplished: i) by forming the section hypergraph $H_S(A,\pi)$, $S = X - \{x\}$, ii) by adding to Y_{ex} a set of new arcs $Y_{\text{ex}}^x \triangleq Y_1^x \cup Y_2^x \cup Y_3^x$

$$Y_1^x \triangleq \{(r,s) | (r,s) \notin Y_{\text{ex}}, (r,x) \in Y_{\text{ex}}, (x,s) \in Y_{\text{ex}}\}$$

$$Y_2^x \triangleq \{(r,s) | (r,s) \notin Y_{\text{ex}}, (r,x) \in Y_{\text{ex}}, s \in \pi_j(Q_n)\}$$

$$Y_3^x \triangleq \{(r,s) | (r,s) \notin Y_{\text{ex}}, (x,s) \in Y_{\text{ex}}, r \in \pi_j(Q_n)\}. \quad \square$$

Definition 4.7: Let $x \in \pi_j(Q_n)$ be a node of $H(A,\pi)$:
i) the external out-degree of x, $d_{\text{ex}}^+(x)$ is

$$d_{\text{ex}}^+(x) = |Y_{\text{ex}}^+(x)|, \quad \text{where } Y_{\text{ex}}^+(x) = \{y \in Y_{\text{ex}} | y = (x,z), z \in Q_n\}$$

ii) the external in-degree of x, $d_{\text{ex}}^-(x)$ is

$$d_{\text{ex}}^-(x) = |Y_{\text{ex}}^-(x)|, \quad \text{where } Y_{\text{ex}}^-(x) = \{y \in Y_{\text{ex}} | y = (z,x), z \in Q_n\}. \quad \square$$

A *self-loop* is an arc $y = (r,s)$ in Y_{ex} such that $r = s$.

As in the standard case [15]–[18], a number of local reduction rules can now be stated, on the basis of which branch-and-bound-type algorithms can easily be set up. Once a MES has been determined, it is a relatively simple task to find a rows and columns permutation such that A is given a bordered block lower triangular form.

Reduction Rule 1: The elimination of $x \in X$ such that

$$d_{\text{ex}}^-(x) d_{\text{ex}}^+(x) = 0$$
$$d_{\text{ex}}^-(x) + d_{\text{ex}}^+(x) \leq 1$$

is index preserving.

Definition 4.8: For any $S \subset X$, let $Ad^+(S) \triangleq \{x | (x,r) \in Y_{\text{ex}}, r \in S\}$ and $Ad^-(S) \triangleq \{x | (r,x) \in Y_{\text{ex}}, r \in S\}$.

Reduction Rule 2: If $|Ad^+(\pi_j(Q_n))| = 1$ and $Ad^+(\pi_j(Q_n)) \cap \pi_j(Q_n) = \varnothing$ or if $|Ad^-(\pi_j(Q_n))| = 1$ and $Ad^-(\pi_j(Q_n)) \cap \pi_j(Q_n) = \varnothing$, then the elimination of all the vertices $x \in \pi_j(Q_n)$ is index preserving.

Definition 4.9: Any $S \subset \pi_j(Q_n)$ is a cover of $Y_{\pi_j(Q_n)}^{\text{ex}}$, where $Y_{\pi_j(Q_n)}^{\text{ex}} \triangleq Y_{\pi_j(Q_n)} \cap Y_{\text{ex}}$, if $(r,s) \in Y_{\pi_j(Q_n)}^{\text{ex}}$ implies $\{r,s\} \cap S \neq \varnothing$.

Definition 4.10: A minimum cover is a cover of minimum cardinality.

Reduction Rule 3: If $Ad^+(\pi_j(Q_n)) \subseteq \pi_j(Q_n)$ or $Ad^-(\pi_j(Q_n)) \subseteq \pi_j(Q_n)$ and if S is a minimum cover of $Y_{\pi_j(Q_n)}^{\text{ex}}$, then at least one MES of $H(A,\pi)$ contains S and no elements of $\pi_j(Q_n) - S$.

Definition 4.11: A doublet is a cycle of length two.

Reduction Rule 4: If, after removing all the arcs forming doublets, $[Ad^+(\pi_j(Q_n)) = \varnothing$ and $Ad^-(\pi_j(Q_n)) \cap \pi_j(Q_n) = \varnothing]$ or $[Ad^-(\pi_j(Q_n)) = \varnothing$ and $Ad^+(\pi_j(Q_n)) \cap \pi_j(Q_n) = \varnothing]$, then the removal from Y of any element of $Ad^-(\pi_j(Q_n)) \times \pi_j(Q_n)$ or $Ad^+(\pi_j(Q_n)) \times \pi_j(Q_n)$, respectively, is index preserving.

Proposition 4.4: If $Ad^+(\pi_j(Q_n)) \cap \pi_j(Q_n) \neq \varnothing$ or $Ad^-(\pi_j(Q_n)) \cap \pi_j(Q_n) \neq \varnothing$, then any MES of $H(A,\pi)$ contains a cover of $Y_{\pi_j(Q_n)}^{\text{ex}}$.

Corollary 4.2 (Reduction Rule 5): If $(x,x) \in Y_{\text{ex}}$, then x belongs to any MES of $H(A,\pi)$.

Reduction Rule 6: If $x \in \pi_j(Q_n)$ and there exists $y \notin \pi_j(Q_n)$ such that i) $(y,r) \in Y_{\text{ex}}$, $\forall r \in \pi_j(Q_n)$, ii) $(x,z) \in Y_{\text{ex}} \Rightarrow (y,z) \in Y_{\text{ex}}$, $\forall z \notin \pi_j(Q_n)$, then the removal of (y,x) is index preserving.

V. The Tearing Algorithm

In this section, a particular way of applying MA to a system $Ax = b$, where A is a bordered block lower triangular matrix, is described and some computational remarks are given.

[1] A problem is said to be hard (NP complete) if it belongs to a class of well-known combinatorial problems (covering, sequencing, knapsack, $0-1$ integer programming, Hamiltonian circuit, etc.) which are equivalent, in the sense that no algorithm terminating with a number of steps bounded by a polynomial in the dimension of the problem (length of the input) exists for their solution. Moreover, it has been shown that a polynomial bounded algorithm for one of them yields polynomial bounded algorithms for all. This result strongly suggests that these problems will remain "intractable" perpetually [14].

Specifically, let

$$A \triangleq \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \in R^{n^2}$$

where

$$A_{11} \triangleq \begin{vmatrix} A_{11}^{11} & 0 & \cdots & 0 \\ A_{21}^{11} & A_{22}^{11} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{q1}^{11} & A_{q2}^{11} & \cdots & A_{qq}^{11} \end{vmatrix} \in R^{(n-m)^2}$$

and, correspondingly, for any $i=0, 1,\cdots,m$, let

$$x^i \triangleq \begin{vmatrix} x_1^i \\ x_2^i \end{vmatrix}, \qquad x_1^i \triangleq \begin{vmatrix} x_{11}^i \\ \cdot \\ \cdot \\ \cdot \\ x_{1q}^i \end{vmatrix} \in R^{n-m}$$

$$h^0 \triangleq b \triangleq \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}, \qquad b_1 \triangleq \begin{vmatrix} b_{11} \\ \cdot \\ \cdot \\ \cdot \\ b_{1q} \end{vmatrix} \in R^{n-m}.$$

Then, it is easy to see that MA can now be given the following form.

Let

$$H \triangleq \begin{vmatrix} A_{12} \\ A_{22}-I_m \end{vmatrix} \triangleq h^1h^2\cdots h^m, \qquad h^i \triangleq \begin{vmatrix} h_1^i \\ h_2^i \end{vmatrix},$$

$$h_1^i \triangleq \begin{vmatrix} h_{11}^i \\ \cdot \\ \cdot \\ \cdot \\ h_{1q}^i \end{vmatrix}, \qquad i=1,2,\cdots m$$

be of rank m and set

$$K \triangleq \begin{vmatrix} 0 \\ I_m \end{vmatrix}$$

then

$$B \triangleq A - HK' = \begin{vmatrix} A_{11} & 0 \\ A_{21} & I_m \end{vmatrix}.$$

Step 1: $i=0$. Compute the LU factorization of A_{kk}^{11}, $k=1, 2,\cdots,q$.

Step 2: $k=0$.

Step 3: $k=k+1$. Compute (by forward elimination and back substitution) x_{1k}^i such that $A_{kk}^{11}x_{1k}^i = h_{1k}^i - \Sigma_{j=1}^{k-1}A_{kj}^{11}x_{1j}^i$.

Step 4: If $k=q$, go to Step 5, otherwise go to Step 3.

Step 5: Compute $x_2^i = h_2^i - A_{21}x_1^i$.

Step 6: $i=i+1$. If $i=m+1$, go to Step 7, otherwise go to Step 2.

Step 7: Compute the LU factorization of I_m+X_2, $X_2 \triangleq |x_2^1x_2^2\cdots x_2^m|$.

Step 8: Compute (by forward elimination and back substitution) x_2 such that $(I_m+X_2)x_2 = x_2^0$.

Step 9: Compute $x_1 = x_1^0 - X_1x_2$, $X_1 \triangleq |x_1^1, x_1^2, \cdots, x_1^m|$.

In order to compare the performances of TA with respect to LU decomposition method the number of multiplications needed in TA is evaluated.

Let $A_{kk}^{11} \in R^{p_k^2}$, ϵ_i^k+1 be the number of nonzero elements in the first row and ξ_i^k+1 be the number of nonzero elements in the first column of the reduced matrix of order p_k-i+1 during the ith step of Gaussian elimination performed in natural order on A_{kk}^{11}. Let N be the total number of nonzero elements in A_{21} and A_{rs}^{11}, $r<s$, $s=2,\cdots,q$. Step 1 of TA requires $\Sigma_{k=1}^q \Sigma_{i=1}^{p_k} (\epsilon_i^k\xi_i^k+\epsilon_i^k)$ multiplications. Step 3 and Step 5 require globally $n-m+(m+1)\cdot(\Sigma_{k=1}^q(\Sigma_{i=1}^{p_k}(\epsilon_i^k+\xi_i^k)+p_k)+N)$ multiplications. Step 7 and Step 8 require $m^3/3-m/3+m^2$ multiplications. Step 9 requires $m(n-m)$ multiplications. Globally, TA requires

$$\sum_{k=1}^q \sum_{i=1}^{p_k} (\epsilon_i^k\xi_i^k+\epsilon_i^k)+(m+1) \cdot \left[\sum_{k=1}^q\left(\sum_{i=1}^{p_k}(\epsilon_i^k+\xi_i^k)+p_k\right)+N\right] + m^3/3+m^2-m/3+(m+1)(n-m)$$

multiplications. LU decomposition requires [10]

$$\sum_{i=1}^{n-1} (\zeta_i\gamma_i+2\zeta_i+\gamma_i)+n \text{ multiplications} \tag{5.1}$$

where ζ_i and γ_i have been defined in Section III.

Assumption 5.1: The elimination orderings in LU decomposition and in TA are the same. Moreover, rows and columns have been rearranged so that the chosen elimination ordering is the natural one. □

In view of Assumption 5.1, let $\eta_i^k = \zeta_i - \epsilon_i^k$ and $\theta_i^k = \gamma_i - \xi_i^k$; then (5.1) can be rewritten as

$$\sum_{k=1}^q \sum_{i=1}^{p_k} (\epsilon_i^k\xi_i^k+\epsilon_i^k) + \sum_{k=1}^q\left(\sum_{i=1}^{p_k}(\epsilon_i^k+\xi_i^k)+p_k\right) + \sum_{k=1}^q \sum_{i=1}^{p_k} (\epsilon_i^k\theta_i^k+\eta_i^k\xi_i^k+\eta_i^k\theta_i^k+2\eta_i^k+\theta_i^k) + \sum_{k=n-m+1}^{n-1} (\zeta_k\gamma_k+2\zeta_k+\gamma_k)+m.$$

TA requires less operations then LU decomposition if the

TABLE II
COMPARISON OF LU DECOMPOSITION (GAUSSIAN ELIMINATION) AND TA

Assumptions	Operations		
	TA	LU	TA/LU
$p >> q,\ m$	$q/3\ p^3$	$(q^2/4+q/12)p^3$	$\frac{4}{1+3q}$
$q >> m,\ p$	$\frac{1}{2}\ p^2(m+1)q^2$	$(p^3/4+(2m+1)p^2/4)q^2$	$\frac{2(m+1)}{p+2m+1}$
$q \approx p = \bar{p} >> m$	$\frac{1}{2}\ (m+4/3)\bar{p}^4$	$\frac{1}{4}\ \bar{p}^5$	$\frac{2(3m+4)}{3\bar{p}}$

following inequality holds:

$$m\left[\sum_{k=1}^{q}\left(\sum_{i=1}^{p_k}(\epsilon_i^k+\xi_i^k)+p_k\right)+N\right]+n+m^3/3+m\left(n-\frac{7}{3}\right)<\sum_{k=1}^{q}\sum_{i=1}^{p_k}(\epsilon_i^k\theta_i^k+\eta_i^k\xi_i^k+\eta_i^k\theta_i^k+2\eta_i^k+\theta_i^k)$$

$$+\sum_{k=n-m+1}^{n-1}(\zeta_k\gamma_k+2\zeta_k+\gamma_k).$$

By inspection, it is possible to claim that TA tends to overcome when m decreases and η_i^k, θ_i^k increase w.r.t. N. A precise evaluation can be given if some particular cases are investigated. In Table II, some comparison are given.

For all the cases considered in Table II, the following assumptions hold: i) $p_k=p$, $k=1,\cdots,q$ and ii) A_{12}, A_{21}, A_{22}, and A_{rs}^{11}, $r\leqslant s$, $s=1,\cdots,q$ are full matrices.

Remark 5.2: If the structure of the system is repetitive, i.e., if the matrices on the main diagonal of A_{11} are equal, TA performs only once Step 1, saving in this way many operations. □

Remark 5.3: TA does not generate any fill-in in A_{rs}^{11}, $r<s$, $s=2,\cdots,q$, and in A_{21}. □

Remark 5.4: At Step 3 of TA at most a square matrix of dimension p_k and a rectangular matrix of dimensions $p_k \times p_r$ (where $p_r=\max_{j=1,\cdots,k-1} p_j$) must be retained contemporarily in the fast memory. This feature makes TA suitable for the analysis of large systems with small computers. □

Remark 5.5: Step 1 of TA can be accomplished by parallel computation. □

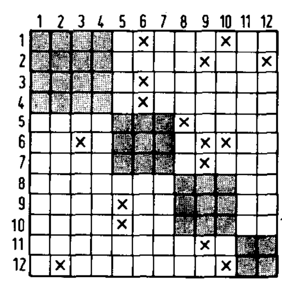

Fig. 2. Zero–nonzero pattern of A.

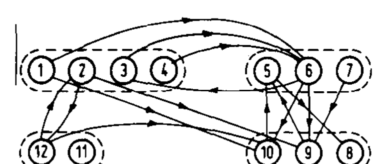

Fig. 3. Hypergraph associated to A.

VI. AN ILLUSTRATIVE EXAMPLE

In order to illustrate most of the definitions and some of the results given in the present paper, a simple example is here worked out in detail.

Fig. 2 shows the zero–nonzero pattern of a (12×12)-matrix A. In this case, A is partitioned according to the regular partition $\bar{\pi}(Q_{12})=\{\{1,2,3,4\},\{5,6,7\},\{8,9,10\},\{11,12\}\}$. The hypergraph $H(A,\bar{\pi})$ associated to A is shown in Fig. 3.

In order to find a MES for the hypergraph of Fig. 3, the local reduction rules presented in Section IV can be used. Fig. 4(a) is obtained by applying to $H(A,\bar{\pi})$ the Reduction Rule 4. The hypergraph of Fig. 4(b) has been ob-

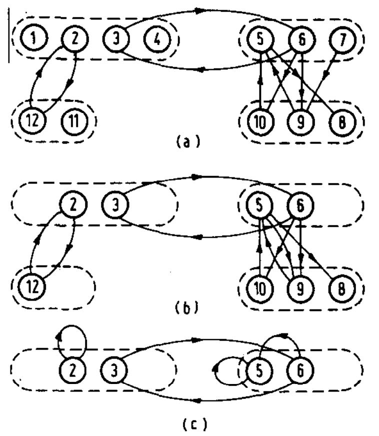

Fig. 4. Step by step reduction of $H(A,\pi)$.

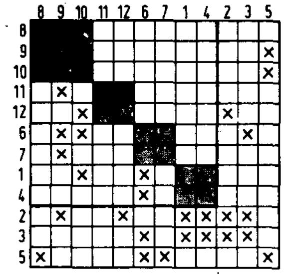

Fig. 5. Optimal bordered lower triangular form of A.

tained from the one of Fig. 4(a) by applying the Reduction Rule 1. Finally, Reduction Rule 2 leads to the hypergraph of Fig. 4(c), where $S=\{2, 3, 5\}$ is apparently a MES as it can also be formally ascertained through the successive (repeated) application of Reduction Rules 5 and 2.

According to Theorem 4.1, A is block S-reducible relative to $\bar{\pi}(\cdot)$. In fact, $\bar{\pi}_S(Q_{12}-S)=\{\{1, 4\}, \{6, 7\}, \{8, 9, 10\}, \{11, 12\}\}$ and there exists a permuation $p^*(\cdot)$, with $p^*(\pi_S)=\{\{8, 9, 10\}, \{11, 12\}, \{6, 7\}, \{1, 4\}\}$, such that $T_{p^*(\pi_S)}(A_S)$ is lower triangular (Fig. 5).

VI. Conclusions

In this paper, tearing methods for solving large scale systems of linear algebraic equations have been discussed. In particular, a modification algorithm which extends previous results given in [10] has been presented and its complexity critically evaluated. Then, the problem of determining what are the "best" elements to be torn in the original given matrix A has been faced. In particular, the reduction of A into a block lower triangular matrix with as few "modifications" as possible has been assumed as goal to achieve. Under reasonable assumptions, the problem of determining the optimal rearranging of A according to the selected goal, has been proven to be equivalent to the determination of a minimum essential set in an hypergraph associated to A. A branch-and-bound algorithm based on a number of local reduction rules has been outlined. Finally, a particular way of applying the previously presented modification algorithm to a matrix rearranged according to the chosen criterion has been discussed and its performances compared with LU decomposition method.

Further work could be done in defining new criteria to be followed in rearranging A and in deriving efficient algorithms to solve the related optimization problems.

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An Efficient Algorithm for Simulation of Transients in Large Power Systems

GEORGE GROSS AND ARTHUR R. BERGEN

Abstract—**The simulation of the transient response of a large interconnected power system involves the solution of a very large system of differential-algebraic equations under a great variety of initial conditions and disturbances. The demands imposed on a digital transient stability program to i) study larger power system interconnections, ii) provide a more detailed representation of the power system components, and iii) permit the simulation of longer time periods, have the effect of increasing the computing time. The importance of, and the need for, efficient computational schemes is apparent. The method presented in this paper makes detailed use of the structural properties of the differential-algebraic system representation. The nonlinear differential-algebraic system is split into a nonstiff part with long time constants coupled to a stiff part with a sparse Jacobian matrix whose longest time constant is shorter than that of the first part. These two parts are linear in their respective states, i.e., the system is semilinear. With the nonstiff part removed, a smaller set of stiff equations with a smaller conditioning number than the original system is obtained. Consequently, longer stepsizes can be used so as to reduce the computation time. The proposed multistep integration schemes exploit the stiffness and semilinearity properties. Numerical results on a small test**

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