Avoiding Zero Pivots in the Modified Nodal Approach

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Abstract—**The modified nodal approach has been widely used for formulating network equations. Although this approach is quite general, zero-diagonal elements may exist in the network matrix. When sparse matrix techniques with *diagonal* pivoting are used to solve these equations, extreme care should be taken so as not to choose a zero-valued pivot. In this paper it is shown that under certain conditions all previously published methods have the potential of generating zero-diagonal pivots, regardless of element values in the network. A simple partitioning and ordering strategy is then presented which guarantees that no zero-valued pivots will be generated for *any* choice of diagonal pivots. The method has been implemented and well tested and various illustrative examples are included.**

I. Introduction

THE MODIFIED nodal approach [1] has been used in many computer-aided circuit analysis programs [2]–[5] for formulating circuit equations. It is well known that although this approach is quite general, zero-diagonal elements may exist in the network matrix. This occurs, for example, when the circuit contains voltage sources, short-circuits, inductors at zero frequency (dc solution) and some types of controlled sources such as current-controlled sources. When sparse matrix techniques with *diagonal* pivoting are used for solving these types of circuit equations, extreme care should be taken so as not to choose a zero-valued pivot. Various methods have been proposed for avoiding pivoting on these zero-diagonal entries. One method involves rearranging and/or combining rows and columns in order to obtain nonzero diagonal elements [1]. Other methods involve ordering the rows and columns with zero-diagonal entries last, in the hope that they will be filled before becoming candidates for pivoting [2], [3]. We show, however, that even if all the zero-diagonal elements which exist in the network matrix at the formulation stage are avoided or filled during the elimination stage, it is possible, under certain conditions, to generate zero-diagonal elements during the Gaussian elimination process regardless of the values of the circuit elements.

From the numerical point of view, it is well known that the solution process will not fail during the solution of linear equations by Gaussian elimination provided that the pivots are all nonzero; however, this will be true only if the *leading* principal minors are all nonzero [6]. It follows then that Gaussian elimination will not fail for *any diagonal* pivoting provided that the leading principal minors for *any* diagonal pivoting are all nonzero. In this regard, we introduce the following definition.

Definition: A matrix M has property F if the leading principal minors of M for any diagonal pivoting are all nonzero.

Although the above definition seems to be somewhat restrictive since a matrix may be factorizable even if it does not have property F it does, however, give complete flexibility in the choice of the diagonal pivot without running the risk of generating a zero pivot during the Gaussian elimination process. As a result, equation reordering for sparsity considerations can be carried out without any restrictions on the order of the variables.

In Section II, we investigate the conditions and derive theorems for when the network matrix in a modified nodal formulations will or will not have property F. We then present a method of partitioning the network variables and performing initial block row interchange in order to produce a network matrix with property F, regardless of element values in the network. Implementation of our method resulted in a modified version of SPICE2 [2]. Using this modified program, many examples which caused computational problems in the original version due to pivoting on zero-diagonal elements were successfully analyzed. Furthermore, our results show that in many cases the number of fills produced by our ordering strategy is much less than that produced by previous ordering strategies, resulting in lower computational cost, and at the same time, more accurate solutions. Some of these examples and results are discussed in Section III. Appendices are included which give details of the network variable partitioning algorithm and proofs of theorems and results presented in the paper.

II. Factorizability Properties of the Modified Nodal Equations

Consider a linear (or linearized) network which is connected; suppose that the branch constitutive equations of the network are given in the following form:

$$\begin{aligned} \boldsymbol{i}_1 &= \boldsymbol{G}_1\boldsymbol{v}_1 + \boldsymbol{H}_1\boldsymbol{i}_2 + \boldsymbol{s}_1 \\ \boldsymbol{v}_2 &= \boldsymbol{H}_2\boldsymbol{v}_1 + \boldsymbol{Z}_2\boldsymbol{i}_2 + \boldsymbol{s}_2. \end{aligned} \tag{1}$$

Equation (1) is very general in the sense that it includes almost all types of element characteristics used in network analysis. Let the topological equations (KCL and KVL) of

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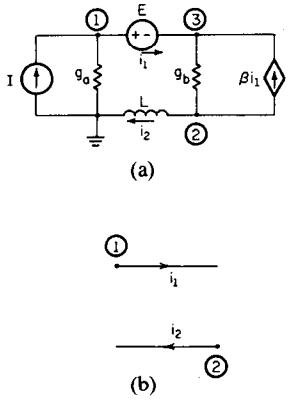


Fig. 1. (a). A network with a cutset of branches whose currents are network variables (see Example 1). (b). The graph G_I of the current branches.

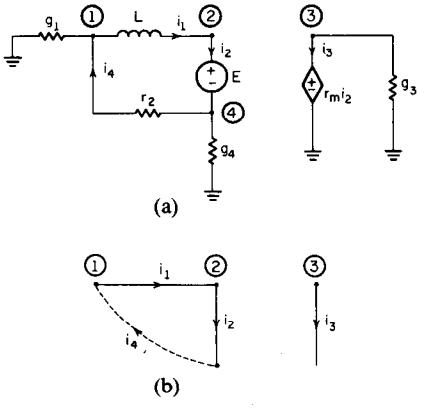


Fig. 2. (a). A network with a loop of branches whose currents are network variables (see Example 2). (b). Graph G_I of the current branches.

the network be partitioned as follows:

$$[A_1 \quad A_2]\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = 0 \tag{2}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix} v_n \tag{3}$$

where $A=[A_1 \quad A_2]$ is the (reduced) incidence matrix, $i=[i_1 \; i_2]^T$ and $v=[v_1 \; v_2]^T$ are the branch currents and voltages, respectively, and v_n is the vector of note-to-datum voltages. Superscript T indicates the transpose.

Eliminating i_1, v_1, and v_2 in (1), (2), and (3), we get the modified nodal equations:

$$\begin{bmatrix} A_1 G_1 A_1^T & A_1 H_1 + A_2 \\ A_2^T - H_2 A_1^T & -Z_2 \end{bmatrix} \begin{bmatrix} v_n \\ i_2 \end{bmatrix} = \begin{bmatrix} -A_1 s_1 \\ s_2 \end{bmatrix}. \tag{4}$$

In (4) Z_2 may contain zero-diagonal elements such as when voltage sources, inductors at zero frequency, and certain types of controlled sources exist in the network. In this case when diagonal pivoting is used, extreme care should be taken so as not to choose a zero-valued pivot. In some ordering strategies [2], [3], the components of i_2 which correspond to the zero-diagonal entries of Z_2 are ordered last in the hope that these zero entries will be filled before they are chosen as pivots. Although this approach avoids pivoting on a zero-diagonal element in Z_2, it may, however, create a zero-valued leading principal minor in $A_1 G_1 A_1^T$ which will cause the Gaussian elimination process to fail. The following example illustrates this case.

Example 1. Consider the network in Fig. 1(a) which contains a cutset of branches whose currents are network variables in a modified nodal formulation. The network matrix in a dc analysis procedure is as follows:

$$\begin{bmatrix} g_a & 0 & 0 & 1 & 0 \\ 0 & g_b & -g_b & \beta & 1 \\ 0 & -g_b & g_b & -1-\beta & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

It is obvious that the matrix in (5) contains a singular principal submatrix.

The above observation can be stated in the form of a theorem.

Theorem 1. Given a linear network having a *cutset* of branches whose currents are declared as network variables in a modified nodal formulation, the network matrix will not have property F regardless of element values in the network.

Proof: In (2) let b_1 represent the branches in the first partition and b_2 the branches in the second partition. Since b_2 contains a cutset and the network is connected, b_1 cannot contain a tree. Hence, A_1 is not of full rank. It follows that $A_1 G_1 A_1^T$ in (3) is singular, regardless of the values in G_1. ■

The following is a network interpretation of Theorem 1: By ordering i_2 last, floating subnetworks are created when b_2 contains a cutset. The admittance matrices of these floating subnetworks which form principal submatrices in the network matrix are singular.

Another approach for avoiding pivoting on the zero-diagonal entries of Z_2 is to interchange the rows in Z_2 containing zero-diagonal entries with the rows of the 'respective' nodal equations.[1] This approach has been briefly mentioned in [1]. It turns out, however, that under certain conditions, interchanging rows of Z_2 with the rows of the respective nodal equations will create zero-valued principal minors, as stated in the following theorem.

Theorem 2. If a linear network has a *loop* of branches whose currents are declared as network variables in the modified nodal formulation, and if the reference node is not contained in the loop, then there is a one-to-one correspondence between the rows in Z_2 corresponding to the branches in the loop and the nodes in the loop which are contained in the reduced incidence matrix. Furthermore, if these rows in Z_2 are interchanged with the rows of their respective node equations, and if there is no coupling

[1] Each row of Z_2 is associated with a branch whose constitutive relation forms that row. A "respective" node of a row of Z_2 is one of the nodes of the associated branch (see also Appendix A on the selection of respective nodes).

among the voltages of the branches in the loop then the resultant network matrix will not have property F regardless of the element values in the network. ■

The proof of the theorem is included in Appendix C. The following example illustrates Theorem 2.

Example 2. Consider the network in Fig. 2(a) which contains a loop of branches whose currents are declared as network variable in a modified nodal formulation. The network matrix of the modified nodal equations in a transient analysis simulation using a backward Euler formula with time step h is as follows:

$$\begin{array}{c} v_1 \quad v_2 \quad v_3 \quad v_4 \quad i_1 \quad i_2 \quad i_3 \quad i_4 \\ \left[\begin{array}{cccc:cccc} g_1 & & & & 1 & & & -1 \\ & 0 & & & -1 & 1 & & \\ & & g_3 & & & & 1 & \\ & & & g_4 & & -1 & & 1 \\ \hdashline 1 & -1 & & & -\frac{L}{h} & & & \\ & 1 & & -1 & & 0 & & \\ & & 1 & & & -r_m & 0 & \\ -1 & & & 1 & & & & -r_2 \end{array}\right] \end{array} \quad (6)$$

Note that due to the cutset formed by i_1 and i_2 at node 2, the admittance submatrix in (6) is singular. Interchanging the rows corresponding to the currents with the rows of the respective node voltages, we get

$$\begin{array}{c} v_1 \quad v_2 \quad v_3 \quad v_4 \quad i_1 \quad i_2 \quad i_3 \quad i_4 \\ \left[\begin{array}{cccc:cccc} 1 & -1 & & & -\frac{L}{h} & & & \\ & 1 & & -1 & & 0 & & \\ & & 1 & & & -r_m & 0 & \\ -1 & & & 1 & & & & -r_2 \\ \hdashline g_1 & & & & 1 & & & -1 \\ & 0 & & & -1 & 1 & & \\ & & g_3 & & & & 1 & \\ & & & g_4 & & -1 & & 1 \end{array}\right] \end{array} \quad (7)$$

It can be easily seen that the upper 4×4 submatrix, as well as the lower 4×4 submatrix, are singular. Thus (7) does not have property F as predicted by Theorem 2.

The method proposed and investigated in this paper is a modification of the method considered in Theorem 2. Assume for the moment that the network does not include any coupled elements and that it contains neither cutsets of current sources and capacitors nor loops of voltage sources, inductors and zero-valued resistors (a zero-valued resistor can also be considered as a voltage source with value 0). Let a graph G_I (possibly disconnected) be first constructed to include those branches in the network whose currents have been declared as network variables, with all the other branches removed (i.e., open-circuited). If G_I contains loops, then a tree (or forest) is chosen, with only finite-valued resistors as links. This is always possible since by assumption no loops of only voltage sources, inductors and zero-valued resistors exist in the circuit. It is well known that the link resistors do not form any cutsets [7].

The network branch currents are then partitioned into three subsets: $\boldsymbol{i}_1$ is the set of currents in the branches of the tree (or forest) of G_I, $\boldsymbol{i}_3$ is the set of currents in the links of G_I and $\boldsymbol{i}_2$ the currents in the remaining branches of the network. The node voltages are partitioned into two subset: $\boldsymbol{v}_{n1}$ consists of the "positive" node voltages of the nodes of the tree (or forest) of G_I, with the exception of one node for each tree in the forest, which could be considered as a reference node for the tree. Note that $\boldsymbol{v}_{n1}$ and $\boldsymbol{i}_1$ have the same dimension. (An algorithm for assigning $\boldsymbol{v}_{n1}$ is given in Appendix A.) $\boldsymbol{v}_{n2}$ is formed by the remaining node voltages in the network. With this partitioning, the KCL and KVL equations can be written as follows:

$$\begin{array}{c} \quad b_1 \quad b_2 \quad b_3 \\ \begin{array}{c} n_1 \\ n_2 \end{array} \left[\begin{array}{ccc} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{array}\right] \left[\begin{array}{c} \boldsymbol{i}_1 \\ \boldsymbol{i}_2 \\ \boldsymbol{i}_3 \end{array}\right] = \boldsymbol{0} \end{array} \quad (8)$$

$$\left[\begin{array}{c} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \end{array}\right] = \left[\begin{array}{cc} \boldsymbol{A}_{11}^T & \boldsymbol{A}_{21}^T \\ \boldsymbol{A}_{12}^T & \boldsymbol{A}_{22}^T \\ \boldsymbol{A}_{13}^T & \boldsymbol{A}_{23}^T \end{array}\right] \left[\begin{array}{c} \boldsymbol{v}_{n1} \\ \boldsymbol{v}_{n2} \end{array}\right]. \quad (9)$$

Let the branch constitutive equations be expressed as follows:

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{Z}_1\boldsymbol{i}_1 + \boldsymbol{s}_1 \\ \boldsymbol{i}_1 &= \boldsymbol{G}_2\boldsymbol{v}_2 + \boldsymbol{s}_2 \\ \boldsymbol{v}_3 &= \boldsymbol{Z}_3\boldsymbol{i}_3 + \boldsymbol{s}_3. \end{aligned} \quad (10)$$

Remark 1. (a) $\boldsymbol{A}_{11}$ in (8) is square and nonsingular with $+1$ on the diagonal and 0 and -1's off the diagonal. (b) $\boldsymbol{A}_{22}$ has rank n_2 since the b_3 branches contain only links of the b_1 tree branches. (c) $\boldsymbol{Z}_1$ in (10) is a diagonal matrix with possibly zeros on the diagonal; while $\boldsymbol{G}_2$ and $\boldsymbol{Z}_3$ are diagonal matrices with positive nonzero entries on the diagonal.

Eliminating $\boldsymbol{i}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ in (8), (9), and (10), we get the modified nodal equations:

$$\left[\begin{array}{cccc} -\boldsymbol{Z}_1 & \boldsymbol{A}_{11}^T & \boldsymbol{A}_{21}^T & \boldsymbol{0} \\ \boldsymbol{A}_{11} & \boldsymbol{A}_{12}\boldsymbol{G}_2\boldsymbol{A}_{12}^T & \boldsymbol{A}_{12}\boldsymbol{G}_2\boldsymbol{A}_{22}^T & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\boldsymbol{G}_2\boldsymbol{A}_{12}^T & \boldsymbol{A}_{22}\boldsymbol{G}_2\boldsymbol{A}_{22}^T & \boldsymbol{A}_{23} \\ \boldsymbol{0} & \boldsymbol{A}_{13}^T & \boldsymbol{A}_{23}^T & -\boldsymbol{Z}_3 \end{array}\right] \left[\begin{array}{c} \boldsymbol{i}_1 \\ \boldsymbol{v}_{n1} \\ \boldsymbol{v}_{n2} \\ \boldsymbol{i}_3 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{s}_1 \\ -\boldsymbol{A}_{12}\boldsymbol{s}_2 \\ -\boldsymbol{A}_{22}\boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{array}\right]. \quad (11)$$

It is obvious that (11) may contain zero-valued principal

minors since Z_1 may have zero-diagonal elements. Interchanging the rows corresponding to i_1 with the rows corresponding to v_{n1} in (11), we get

$$\begin{bmatrix} A_{11} & A_{12}G_2A_{12}^T & A_{12}G_2A_{22}^T & A_{13} \\ -Z_1 & A_{11}^T & A_{21}^T & 0 \\ A_{21} & A_{22}G_2A_{12}^T & A_{22}G_2A_{22}^T & A_{23} \\ 0 & A_{13}^T & A_{23}^T & -Z_3 \end{bmatrix} \begin{bmatrix} i_1 \\ v_{n1} \\ v_{n2} \\ i_3 \end{bmatrix} = \begin{bmatrix} -A_{12}s_2 \\ s_1 \\ -A_{22}s_2 \\ s_3 \end{bmatrix}. \quad (12)$$

Theorem 3. For any positive diagonal matrices G_2 and Z_3 the matrix in (12) will have property F. ■

The proof is included in Appendix C. Note that the theorem can also be considered as a theorem on the existence of a solution to the network equations since it guarantees that the determinant of the network matrix is nonzero. To illustrate the theorem we reconsider Examples 1 and 2. Referring to the circuit in Fig. 1(a) which is the one used in Example 1, the graph G_I of the circuit is shown in Fig. 1(b). Nodes 1 and 2 are chosen as the nodes of the trees in G_I. The modified nodal matrix with the circuit variables partitioned as in (11) is as follows:

$$\begin{array}{c} \begin{matrix} i_1 & i_2 & v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & g_a & 0 & 0 \\ \beta & 1 & 0 & g_b & -g_b \\ -1-\beta & 0 & 0 & -g_b & g_b \end{bmatrix} \end{array}. \quad (13)$$

Interchanging the rows corresponding to v_1 and v_2 (the nodes of G_I) with the rows of i_1 and i_2, we get the following matrix:

$$\begin{array}{c} \begin{matrix} i_1 & i_2 & v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & g_a & 0 & 0 \\ \beta & 1 & 0 & g_b & -g_b \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -1-\beta & 0 & 0 & -g_b & g_b \end{bmatrix} \end{array} \quad (14)$$

which has property F.

Consider now the circuit shown in Fig. 2(a) which has been analyzed in Example 2. The graph G_I of the circuit is shown in Fig. 2(b), where i_1, i_2, and i_3 are the currents in the tree branches of G_I, and i_4 is the current in the link. Interchanging the rows corresponding to i_1, i_2, and i_3 with the rows of their respective node voltages v_1, v_2, and v_3 in (6), we get the following matrix:

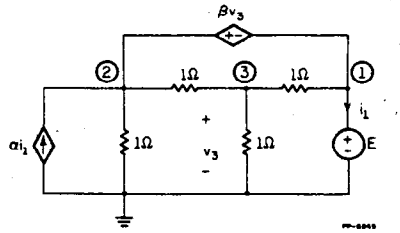


Fig. 3. A network with controlled sources which results in a zero-valued pivot for specific values of controlled source parameters (see Example 3).

$$\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & i_1 & i_2 & i_3 & i_4 \end{matrix} \\ \left[\begin{array}{cccc|cccc} 1 & -1 & & & -\frac{L}{h} & & & \\ & 1 & & -1 & & 0 & & \\ & & 1 & & & -r_m & 0 & \\ & & & g_4 & & -1 & & 1 \\ \hline g_1 & & & & 1 & & & -1 \\ & 0 & & & -1 & 1 & & \\ & & g_3 & & & & 1 & \\ -1 & & & 1 & & & & -r_2 \end{array}\right] \end{array} \quad (15)$$

which has property F for all positive values of g_1, g_3, g_4, r_2, and r_m. Note that the matrix need not be arranged as in (11) before the row interchange is carried out.

The zero-valued principal minors that have been avoided in Theorem 3 are caused mainly by two factors: (a) zero-valued entries in Z_1 due to element values, and (b) topological reasons (loops and cutsets of current variables) that create zero-valued principal minors regardless of element types and values. These zero-valued principal minors are avoided by performing row interchanges. However, controlled sources are not included in Theorem 3. When controlled sources exist in the network the matrix in (12) may contain zero-valued principal minors for *specific* element values, as shown in the following example. However, this situation only occurs for carefully chosen element connections and values, and it is very unlikely that it will occur in the simulation of practical circuits.

Example 3. Consider the network in Fig. 3 which contains controlled sources. Following the ordering techniques proposed in Theorem 3, the matrix below is obtained *after* the row interchange has been carried out:

$$\begin{array}{c} \begin{matrix} i_1 & i_2 & v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & -1 \\ -\alpha & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ -\beta & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 3 \end{bmatrix} \end{array} \quad (16)$$

the matrix in (16) has property F except when $\alpha = 1$.

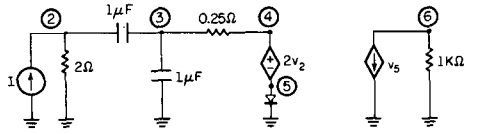


Fig. 4. Circuit used in Example 4.

TABLE I

Node	2	3	4	5	6
Modified SPICE2 node voltages	2.000	4.000	4.000	0.0000	0.0000
Original SPICE2 node voltages	2.000	2.3240	2.3240	-1.6760	1675.9999

III. Implementation

The implementation of the method proposed in Theorem 3 has resulted in a modified version of SPICE2 [2]. In this modified version the "positive" nodes of the branches of G_I of a given circuit are first determined using the algorithm given in Appendix A. The network matrix is constructed using the element stamps as in [1]. The row interchange proposed in (12) is done by a simple change in the pointer system; no physical exchange of row entries is made. The sparse matrix reordering is then carried out using the Markowitz criterion [8] with diagonal pivoting.

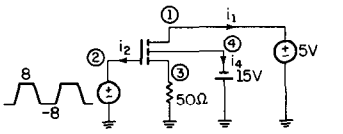
Examples which caused computational problems in the original version of the program due to pivoting on zero diagonal elements were successfully analyzed using this modified version.[2] Furthermore, the results we obtained show that in many cases the number of fills produced by our ordering strategy is far lower than that produced by previous ordering strategies, resulting in less computational cost, and at the same time, more accurate solutions. We present here a small selection of the examples analyzed by the modified version and compare the results with those obtained by the original version.

Example 4. The circuit shown in Fig. 4 was analyzed using both the original and the modified SPICE2 program. The results of the dc analysis are shown in Table I.

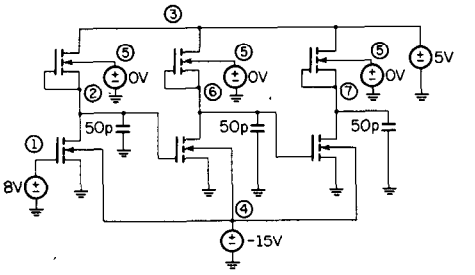
In this circuit the diode is reverse biased. The equivalent resistance used in SPICE2 for this diode is $0.721\times10^{12}\ \Omega$, as a result the computed i_1 in SPICE2 is 2.324×10^{-12} A, instead of the correct value, which should be 0.0 A. This inaccuracy in computing i_1 makes $v_5=-1.6760$ V and $v_6=1675.9999$ V instead of 0.0 V.

Example 5. The two circuits shown in Fig. 5(a) and (b) were also analyzed using both the original and the modified versions of SPICE2. The CPU times required by the equation solving subroutines in both programs for both circuits are given in Table II.

[2] By the original version of SPICE2 we mean the version of vintage 1975. More recent versions of SPICE avoid the computational problems which could be caused by pivoting on zero-diagonal elements by checking the value of the pivot during the solution process and repivoting if necessary [12].

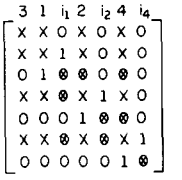


(a)

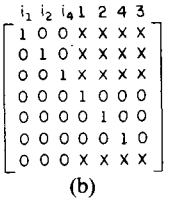


(b)

Fig. 5. (a), (b) Circuits used in Example 5(a) and (b). SPICE2 transistor parameter values are: VTO=0.5, BETA=0.5, GAMMA=0.5, CGS=0.1P, CGD=0.1P, CGB=0.1P, CBD=0.1P, CBS=0.1P.



(a)



(b)

Fig. 6. (a) Structure of the network matrix for the circuit in Fig. 5(a) as formulated by the original version of SPICE2 (⊗ indicates a fill). (b) Structure of the network matrix for the circuit in Fig. 5(a) as formulated by the modified version of SPICE2.

TABLE II

Circuit		CPU time for the equation solving subroutine	number of variables	number of operations per iteration
5(a)	modified SPICE2	0.9090 sec.	7	16
	original SPICE2	1.9740 sec	7	71
5(b)	modified SPICE2	0.031 sec	10	30
	original SPICE2	0.108 sec	10	101

The difference in the number of operations between modified SPICE2 and the original SPICE2 in Table II can be explained as follows: In the original SPICE2, the matrix formulated by the modified nodal approach for the circuit in Fig. 5(a) is as shown in Fig. 6(a). It can be seen that although the number of off-diagonal elements of the rows

and columns corresponding to i_1, i_2, and i_4 is small, they are not chosen as pivots until their corresponding zero-diagonal entries are filled. This delay causes the number of fills to increase greatly. In our approach, the matrix formulated for the circuit in Fig. 5(a) is as shown in Fig. 6(b). It can be seen that the number of fills is now zero, and consequently, the number of operations is reduces.

IV. Discussion

In this paper we have investigated the factorizability properties of the modified nodal equations used in network analysis. We derived conditions for which the equation matrix is or is not factorizable for any diagonal pivot selection. We then presented a strategy for partitioning the network variables and performing initial block row interchange which resulted in a matrix ((12)) factorizable for any diagonal pivoting (Theorem 3), provided the element values are positive and no coupled elements exist in the network. When negative valued elements and coupled elements exist in the network, the result still applies for *almost* all values for the rest of the elements.

In (12) the leading principal submatrix, A_{11} contains $+1$'s on the diagonal and 0 and -1's off the diagonal. If i_1 is eliminated first, then only row additions are involved and no multiplications or divisions are necessary (see Appendix B). Similarly, if $Z_1 = 0$ and both i_1 and v_{n1} are eliminated first, only row additions are involved. Note that for Theorem 3 to apply it is not necessary to have all the tree branches of G_f included in Z_1. It is possible to have Z_1 include only those branches in G_f that contribute zero-diagonal entries provided that they do not form loops, and the remaining branches can be included in Z_3 as long as these branches do not form cutsets.

Although by initially performing row interchange (which is equivalent to off-diagonal pivoting), the network matrix loses some of its symmetry, this, however, is not a drawback. In fact we have observed, in many of the examples we have analyzed, that by using this initial off-diagonal pivoting, the number of fills is much lower than that produced by strictly diagonal pivoting.

Our method has been implemented in a modified version of SPICE2 with little change to the program. A simple subroutine was added to assign i_1 and v_{n1} as described in Appendix A, and the block row interchange needed in (12) was accomplished by a simple change in the pointer system.

Finally, it should be pointed out that if one uses nodal analysis, rather than the modified nodal analysis, one could prove diagonal dominance, and hence property F, when the network contains positive-valued uncoupled resistors only [7]. When coupled or negative-valued elements are allowed, it is then possible that for certain sets of element values numerical instability may occur. The safest way to guard against such numerical instability in the most general case is to perform (partial or complete) pivoting during the solution process of the equations. This approach, however, would be rather costly in circuit analysis, especially when sparse matrix techniques are being used, since the pivot order may have to be changed at every iteration. Our purpose in this paper has been to develop a method of ordering the circuit variables and the equations at a preprocessing stage from graphical considerations only, without a knowledge of the actual values of the elements in the circuit. The aim is to prevent the creation of and pivoting on zero-diagonal elements which would otherwise be created for any element values.

Appendix A

Positive Node Selection Algorithm

Let p be a vector of dimension n, where n is the number of ungrounded nodes in a given network. Let p_j be the number of branches whose currents belong to i_1 in (10) and are incident at node j. Note that p_j can be zero or a positive integer. In the following steps, whenever node j of a current branch is chosen as positive, p_j is reduced to zero and p_k at its "negative" node k is reduced by one.

(1) The ungrounded nodes of all grounded current branches belonging to i_1 are selected as positive first.

(2) If p_k of node k of a current branch is one, then node k is selected positive for that particular current branch. If more than one node have their p_k values equal to one and if some of these nodes do not have a conductance (i.e., a resistance whose current is not a circuit variable) connected to them, then one of these nodes is chosen positive first. Otherwise, any one of the nodes that has its p_k value equal to one is chosen positive.

Step (2) is repeated until all the branches corresponding to i_1 have been processed. Note that up to this point there is always at least one node whose p_k value is one. This is because the branches corresponding to i_1 do not form loops. Note also that the number of positive nodes is equal to the number of elements in i_1. The polarities of the currents in the current branches are associated with the positive node assignments.

Appendix B

Let G_d be the *directed* graph of a tree or a forest. Let one node in each tree of the forest be designated as a reference node. Let A_d be the *reduced* incidence matrix of G_d with the nodes and branches arranged in such a way that A_d has $+1$'s on the diagonal (Appendix A describes one way of assigning the nodes and the directions in the branches so that A_d has $+1$'s on its diagonal). It is clear that A_d is square and nonsingular.

Lemma B.1. For any diagonal pivoting the LU factors A_d contain, 0, $+1$ and -1 entries only, with the diagonal entries of both L and U being $+1$'s and the off-diagonal entries being 0 or -1's.

Proof: It is obvious that A_d itself contains 0, $+1$, or -1 entries only. Let A_d be formulated with current i_k

chosen as the first pivot where i_k flows in branch b_k, which is connected between nodes i and j. After row and column interchange the first row and column A_d will have the following form:

$$A_d = \begin{array}{c} \\ j \end{array} \begin{array}{c} \begin{array}{cccc} 1 & a_{12} & 0 & a_{1n} \end{array} \\ \left[\begin{array}{cccc} -1 & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \end{array}$$

where node j is assumed to be in G_d; otherwise column one would be all zeros below the diagonal. Note that entry $a_{ij}=0$ because G_d does not have any loops and a_{1i} $i=2,3,\cdots,n$ are either zero or -1. Pivoting on a_{11} amounts simply to adding row 1 to row j. Since adding any two rows in the incidence matrix of a directed graph produces a row with 0, $+1$, or -1 entries, row j will then contain 0 and -1's with $+1$ on the diagonal. Let the submatrix generated by pivoting on a_{11} be denoted by $\hat{A}_d$. $\hat{A}_d$ can be considered as the incidence matrix of a directed graph $\hat{G}_d$ where $\hat{G}_d$ is derived from G_d by removing branch b_k and merging node 1 with node j [9]. Thus $\hat{A}_d$ has the same properties as A_d, and pivoting on its first diagonal entry will produce a submatrix with ones on the diagonal and 0 and -1's elsewhere. This proves the lemma.

Appendix C

Proof of Theorem 2

Without loss of generality, let all the branches b_2 whose currents are declared as network variables form one loop. Since we have assumed that the reference node is not contained in the loop, then from Appendix A there is a respective node for each branch in the loop. Let the nodes of the loop be denoted by n_1 and the remaining nodes of the network by n_2. Partitioning the KCL and KVL equations, we get

$$\begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \mathbf{0} \end{bmatrix} \end{array} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \mathbf{0} \tag{C.1}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \end{bmatrix} \tag{C.2}$$

while the branch constitutive equations are given as

$$\begin{aligned} i_1 &= G_1 v_1 + H_1 i_2 + s_1 \\ v_2 &= H_2 v_{n2} + Z_2 i_2 + s_2. \end{aligned} \tag{C.3}$$

Note that in (C.1) $A_{22}=\mathbf{0}$ since branches b_2 are incident with nodes n_1 only. In (C.1) $n_1=b_2$, and it follows that each column in A_{12} contains exactly two nonzero entries, $+1$ and -1 and, therefore, A_{12} is singular. Eliminating i_1, v_1, and v_2 from (C.1), (C.2), and (C.3) we get the modified nodal equations:

$$\begin{bmatrix} A_{11}G_1A_{11}^T & A_{11}G_1A_{21}^T & A_{11}H_1+A_{12} \\ A_{21}G_1A_{11}^T & A_{21}G_1A_{21}^T & A_{21}H_1 \\ A_{12}^T & -H_2 & -Z_2 \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ i_2 \end{bmatrix} = \begin{bmatrix} -A_{11}s_1 \\ -A_{21}s_1 \\ s_2 \end{bmatrix}. \tag{C.4}$$

Interchanging the rows corresponding to i_2 with the rows of v_{n1} in (C.4), we get

$$\begin{bmatrix} A_{12}^T & -H_2 & -Z_2 \\ A_{21}G_1A_{11}^T & A_{21}G_1A_{21}^T & A_{21}H_1 \\ A_{11}G_1A_{11}^T & A_{11}G_1A_{21}^T & A_{11}H_1+A_{12} \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ i_2 \end{bmatrix} = \begin{bmatrix} -A_{11}s_1 \\ -A_{21}s_1 \\ s_2 \end{bmatrix}. \tag{C.5}$$

Since A_{12} is singular, a zero-valued principal minor exists in (C.5). This completes the proof.

Proof of Theorem 3

First, we introduce some definitions. Let $a=\{1,2,\cdots,p\}$ and $b=\{1,2,\cdots,q\}$ be two sets of integers. Let $A(ab)$ denote a submatrix of a given matrix A which is formed by the intersection of the rows and columns of A corresponding to sets a and b, respectively. $A(a.)$ represents a submatrix formed by the rows of A corresponding to set a.

Consider now the partitioned incidence matrix given in (8) in Section II:

$$\begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} & A_{11}^{-1}A_{13} \\ \mathbf{0} & A_{22}-A_{21}A_{11}^{-1}A_{12} & A_{23}-A_{21}A_{11}^{-1}A_{13} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & B_{12} & B_{13} \\ \mathbf{0} & A'_{22} & A'_{23} \end{bmatrix}. \tag{C.6}$$

A graph-theoretic interpretation of (C.6) is as follows:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix}$$

represents the incidence matrix of a subgraph formed by those branches that are incident at nodes n_1, where b_1 is a tree (or a forest) of this subgraph. It follows that

$$\begin{bmatrix} I & A_{11}^{-1}A_{12} & A_{11}^{-1}A_{13} \end{bmatrix} \equiv \begin{bmatrix} I & B_{12} & B_{13} \end{bmatrix}$$

is a fundamental cutset matrix of the subgraph [10]. The submatrix $[A'_{22} \quad A'_{23}]$ in (C.6) can be interpreted as the incidence matrix of a graph formed by removing branches

b_1 from the original network and collapsing nodes n_1.

Let us now consider the network matrix defined in (12). A_{11} has property F since it is the incidence matrix of a tree with $+1$'s on the diagonal; Z_3 has property F since it is diagonal with nonzero diagonal elements; and $A_{22}G_2A_{22}^T$ has property F, because A_{22} is of full rank (see Remark 1(b) in Section II) and G_2 is a diagonal matrix with positive nonzero diagonal elements [7]. Suppose we now choose any principal submatrix of the matrix in (12) and rearrange the submatrix as follows:

$$\begin{bmatrix} A_{11}(aa) & A_{12}(a.)G_2A_{12}^T(b.) & A_{12}(a.)G_2A_{22}^T(c.) & A_{13}(ad) \\ -Z_1(ba) & A_{11}^T(bb) & A_{21}^T(cb) & 0 \\ A_{21}(ca) & A_{22}(c.)G_2A_{12}^T(b.) & A_{22}(c.)G_2A_{22}^T(c.) & A_{23}(cd) \\ 0 & A_{13}^T(bd) & A_{23}^T(cd) & -Z_3(dd) \end{bmatrix} \tag{C.7}$$

where a, b, c, and d represent the sets of indexes of the rows and columns chosen from the various submatrices of (12).

The determinant of the matrix in (C.7) can be written as

$$\det A_{11}(aa)\det\left\{\begin{bmatrix} I & Z_1(ba)B_{12}(a.) & Z_1(ba)B_{13}(ad) \\ 0 & A'_{22}(c.) & A'_{23}(cd) \end{bmatrix} \cdot \begin{bmatrix} I & & \\ & G_2 & \\ & & Z_3^{-1}(dd) \end{bmatrix} \begin{bmatrix} I & 0 \\ B_{12}^T(b.) & A'^T_{22}(c.) \\ B_{13}^T(bd) & A'^T_{23}(cd) \end{bmatrix}\right\} \cdot \det \begin{bmatrix} A_{11}^T(bb) & A_{21}^T(cb) \\ 0 & I \end{bmatrix} \det\left[-Z_3(dd)\right]. \tag{C.8}$$

It is obvious that

$$\det A_{11}(aa) \neq 0$$

and

$$\det \begin{bmatrix} A_{11}^T(bb) & A_{21}^T(cb) \\ 0 & I \end{bmatrix} \neq 0.$$

Suppose $b \leq a$, then $Z_1(ba) \equiv [Z_1(bb) \vdots 0]$, where $Z_1(bb)$ is diagonal. It follows that

$$\left[Z_1(ba)B_{12}(a.) \quad Z_1(ba)B_{13}(ad)\right] = \left[Z_1(bb)B_{12}(b.) \quad Z_1(bb)B_{13}(bd)\right]. \tag{C.9}$$

Substituting (C.9) in (C.8), we get

$$\det\left\{\begin{bmatrix} I & Z_1(bb)B_{12}(b.) & Z_1(ba)B_{13}(bd) \\ 0 & A'_{22}(c.) & A'_{23}(cd) \end{bmatrix} \cdot \begin{bmatrix} I & & \\ & G_2 & \\ & & Z_3^{-1}(dd) \end{bmatrix} \begin{bmatrix} I & 0 \\ B_{12}^T(b.) & A'^T_{22}(c.) \\ B_{13}^T(bd) & A'^T_{23}(cd) \end{bmatrix}\right\}. \tag{C.10}$$

Using the Binet–Cauchy theorem,[3] the determinant in (C.10) can be shown to be nonzero.

If $b > a$, then

$$Z_1(ba) \equiv \begin{bmatrix} Z_1(aa) \\ 0 \end{bmatrix},$$

where $Z_1(aa)$ is diagonal. Again using the Binet–Cauchy theorem, the determinant of the triple product in (C.8) can be shown to be nonzero.

This completes the proof.

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[3] The *Binet–Cauchy theorem* states as follows [11]: Given the product of two matrices A B, where A is $n \times m$ and B is $m \times n$ (assume $n \leq m$); then $\det [A \quad B] = \Sigma$ (major determinant of A x corresponding major determinant of B) where Σ means summation "for all possible." A major determinant of A is the determinant of an n by n submatrix of A. If the submatrix of A is formed by taking columns $j_1, j_2, \cdots, j_n$ of A, then the *corresponding* major determinant of B is the determinant of a submatrix of B formed by rows $j_1, j_2, \cdots, j_n$.

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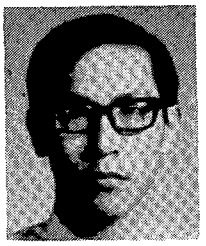

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✤

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Frequency-Domain Considerations of LSV Digital Filters

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Abstract— **The present paper develops a framework for the analysis and synthesis of linear shift-variant (LSV) digital filters in the frequency domain. First, LSV digital filters are theoretically modeled by the successive use of linear shift-invariant (LSI) filters. On the basis of the model, we present an interpretation of shift-variant spectral modification or filtering. Further, shift-variant digital filtering is discussed in relation to the notions of the short-time spectrum and the generalized frequency function. In addition, we propose an efficient implementation procedure which reduces the number of filter coefficients and the amount of computation. The effectiveness of LSV digital filters in processing time-varying signals is demonstrated by experimental verification.**

I. INTRODUCTION

LINEAR shift-invariant (LSI) digital filters have become important tools in a multitude of diverse fields of science and technology. Often, the use of LSI digital filters is insufficient to process various kinds of signals. In seismic data processing, for example, linear shift-variant (LSV) digital filters have been extensively used [1]–[4]. Thus it is of practical and theoretical interest to study LSV digital filters [1]–[7].

Two of the most important applications of digital filters are system identification and modeling [8] and spectral modification [9]. In system identification, the objective is to find some parameters such as impulse responses and coefficients of difference equations to simulate the characteristics of practical systems. The representation of the system in terms of an LSI digital filter follows directly from the assumed stationarity of the system. In many applications, however, LSV digital filters may offer a more accurate representation of the system because of the presence of nonstationary components in some practical systems. As an example, LSV digital filters have been used to model the vocal tract in a speech analysis and synthesis system [10].

Alternatively, the objective of spectral modification or filtering is directed toward removing interference such as

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