Diakoptics—The Solution of System Problems by Tearing

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Invited Paper

Abstract—A piecewise procedure called diakoptics is described. An overview of the theory is presented, and a summary of the applications that have been carried out to date in the power industry has been included. This paper is not a mere recapitualtion, but contains a number of new ideas which have not been presented previously. Only relevant mathematics have been included, and derivations or detailed mathematical presentations that appear elsewhere have been referenced. The paper starts with a historical review in which the classic problem through which diakoptics was conceived is presented. The tearing cases considered include torn subdivisions radially attached as well as torn subdivisions not attached. Only the most basic cases are considered for brevity and clarity of presentation.

Introduction

THE BASIC IDEA of diakoptics is to solve a large system by breaking or tearing it apart into smaller subsystems; to first solve the individual parts, and then to combine and modify the solutions of the torn parts to yield the solution of the original untorn problem. The result of the procedure is identical to one that would have been obtained if the system had been solved as one.

Consider Fig. 1 which may represent a large power system. The solution, through diakoptics, is obtained by tearing the system apart and to solve each part separately (A, B, C, and D) with no contribution from the neighboring parts considered. The contribution to the total solution, due to the interconnections of the torn parts, is considered separately.

The torn parts or subdivisions often occur quite naturally, and thus also the corresponding lines of tear, as for example, the boundaries between power companies.

The uses of diakoptics are at least twofold: in the first application, larger systems can be solved efficiently by the use of diakoptics on a given computer than would otherwise be possible by processing the torn parts through the computer serially. The second application employs a multiplicity of computers which essentially operate in parallel, and thus provide more speed of execution than by the use of a single computer. The computers can be physically next to each other, thus forming a cluster of computers, or they can be miles apart. Each computer in the latter application can work on the solution of a given part. In the case of Fig. 1, four computers would be employed. Extra computational capability is required to calculate the contribution to the total solution due to the interconnections of the torn parts, which can be provided by the computers working on the torn parts, or by extra computers provided for that purpose. A summary of specific applications with corresponding references is presented in the paper.

Both applications above involve conventional computers, which operate more or less serially. Future computers may well have parallel computing capability, similar to the second application, in which case we can expect larger problems to be solved with greater speeds by the use of diakoptics than by solving the problem by conventional methods.

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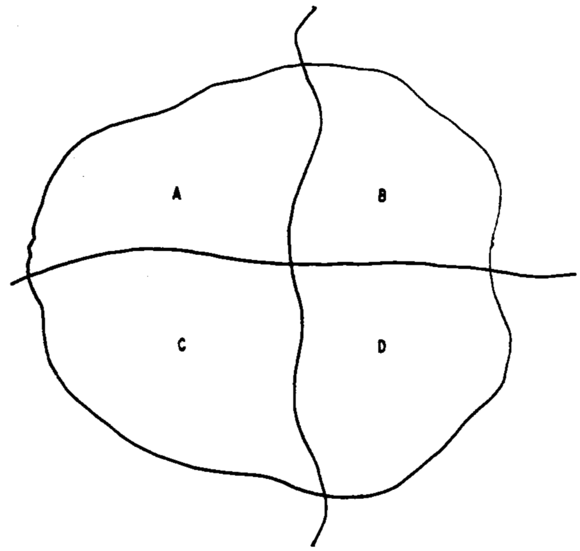


Fig. 1. System torn into four parts.

Historical Review

The word diakoptics comes from the Greek words "kopto" meaning to break or to tear apart, and "dia" which reinforces the word to follow as the English word "very." Diakoptics was conceived by the late Gabriel Kron in the early 1950's, with the word diakoptics coined by the late Prof. Stanley of the Department of Philosophy of Union College, Schenectady, N. Y. [1], [2]. As in many other discoveries, diakoptics was found unintentionally when solving an engineering problem.

The problem was how to obtain the total losses of a large interconnected power system, given the loss models of the individual power companies. Consider Fig. 2 as an example; it represents an interconnected power system consisting of three individual power companies (A, B, and C). The lines between the companies represent interconnecting tie lines. The models by which the real losses (P_L) of the individual companies can be calculated take the form of a square matrix, and each one is of order equal to the number of generators contained in the area plus the number of ties that connect the area to the rest of the system.

The losses of area A, for example, are given by

$$P_{LA} = P_t^A B_{AA} P^A \tag{1}$$

where P^A is the power vector of area A and contains generator and load powers P^{GA} and tie power P^{TA}; t stands for transpose in (1) and B_{AA} represents the loss model of area A.

The composite matrix of the loss models of all three areas

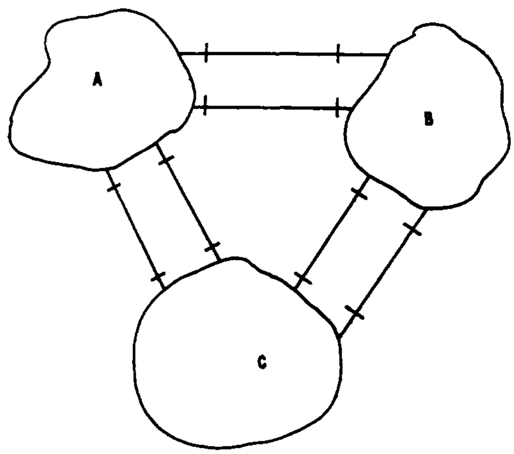


Fig. 2. Example of interconnected power system consisting of three power companies (A, B, C).

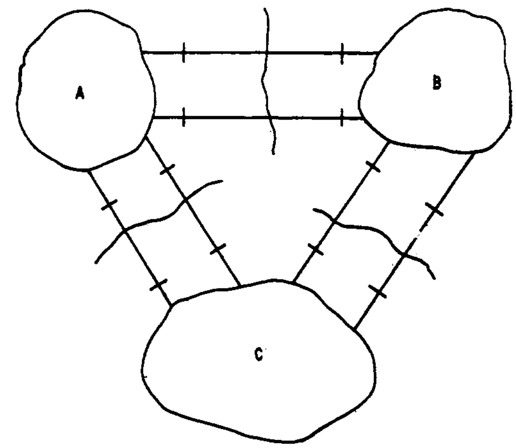


Fig. 3. Interconnected power system torn into three parts.

for Fig. 2 is as follows:

$$B_{\alpha\beta} = \begin{array}{c|ccc|c} & GA\ TA & GB\ TB & GC\ TC & \cdots \\ \hline GA,\ TA & B_{AA} & & & \\ GB,\ TB & & B_{BB} & & \\ GC,\ TC & & & B_{CC} & \\ \vdots & & & & \ddots \end{array} \qquad P^{\alpha} = \begin{array}{c|c} GA & P^{GA} \\ TA & P^{TA} \\ GB & P^{GB} \\ TB & P^{TB} \\ GC & P^{GC} \\ TC & P^{TC} \\ & \vdots \end{array} \tag{2}$$

GA, GB, GC represent generator and load axes, and TA, TB, TC represent tie axes of the areas.

The losses of the total system similar to (1) are

$$P_L = P_t{}^{\alpha} B_{\alpha\beta} P^{\beta}, \qquad \alpha, \beta = 1, 2 \cdots. \tag{3}$$

Note that (3) can only be executed if the tie powers (P^{TA}, P^{TB}, P^{TC}), representing flows between the areas in Fig. 2 are known. But these flows are generally not known, which is the crux of the classic diakoptics problem: specifically, the problem is how to obtain the total solution of a system, such as that in Fig. 2, from the solution of its parts. Note that (2) can be interpreted to represent the solution of the torn system in Fig. 3; (2) represents the solution of the system in Fig. 2 which is shown as torn into three parts or areas. The problem then is how to change or modify the area or torn solutions of Fig. 3 so that they apply to the untorn original problem in Fig. 2.

It should be clear that the interconnections between the subdivisions are not represented in (2) and modeling is therefore required. This is precisely what Kron accomplished through a series of contour (open- and closed-path) transformations which define the interconnections and link the subdivisions [2], [3].

The preceding problem in the 1950's, in which a total solution was obtained from the solution of its parts, triggered research starting in the 1950's to the present [3]–[38].

This author had a close association with Kron for over a decade, starting in the late 1950's, and with Kron's encouragement, evolved a theory underlying diakoptics which can be called the "Contour Theory of Networks." It utilizes open- and closed-path contours and equation structures of currents and voltages associated with the contours. The closed-path or mesh contours are well known to engineers from the time of Maxwell, but not the open-path contours. The latter are the more correct duals of the Maxwell mesh contours, and trace out a network such that the conventional junction pairs of network theory define its endpoints; there are as many independent open paths as there are junction pairs in a network. A nonsingular transformation matrix, which is a branch to contour matrix defined by the contours, is central to the theory. A detailed presentation of the theory is given in [3].

Review of Material Covered

This paper in the main will review two tearing cases as follows [21]:

1) torn subdivisions radially attached;
2) torn subdivisions not attached.

In the first case, the lines of tear must be such that the subdivisions form a radial network when the interconnecting branches are removed. Fig. 4 illustrates this case.

A special but important case of subdivisions radially attached is the case where all subdivisions emanate from a single common bus which often is ground as shown in Fig. 5.

The second tearing case is one where the torn subdivisions are not attached, which is the case previously illustrated in Fig. 3. The classic problem previously discussed is in the second category, and the solution to the problem will be outlined at the conclusion of the material of the second tearing case.

A synopsis of the pertinent theory in each case is first presented, followed by the piecewise algorithm. The theory is important, since it illustrates what the algorithms are accomplishing, and defines the elements used in the algorithms. Applications of the algorithms are summarized at the end of the paper. For the treatment of the dual cases, the reader is referred to [1], [3], and [12].

Torn Subdivisions Radially Attached

Subdivision Level

An example network of the type shown in Fig. 5 with single bus common will be used for illustration and appears in Fig. 6. The material equally applies to the case with multiple common buses illustrated in Fig. 4 except where specifically noted.

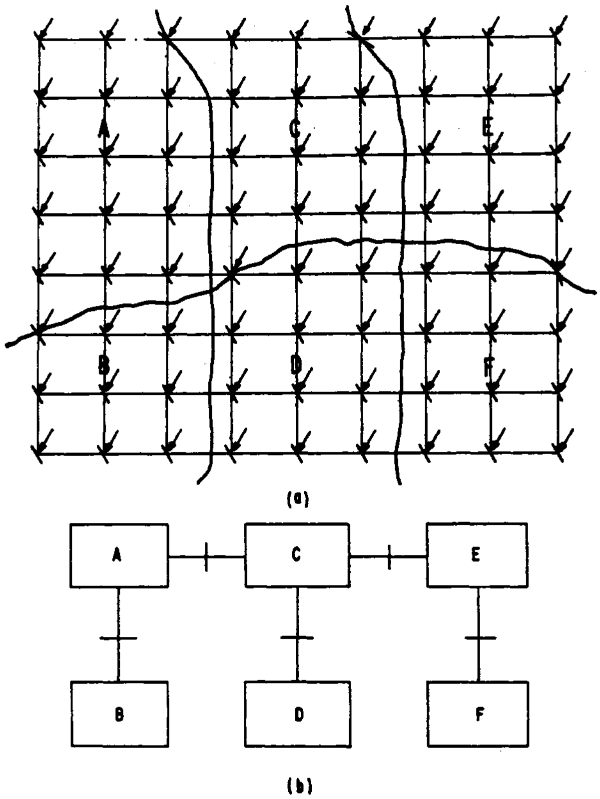


Fig. 4. (a) Network with lines of tear indicated. (b) Subdivisions radially attached—multiple common buses.

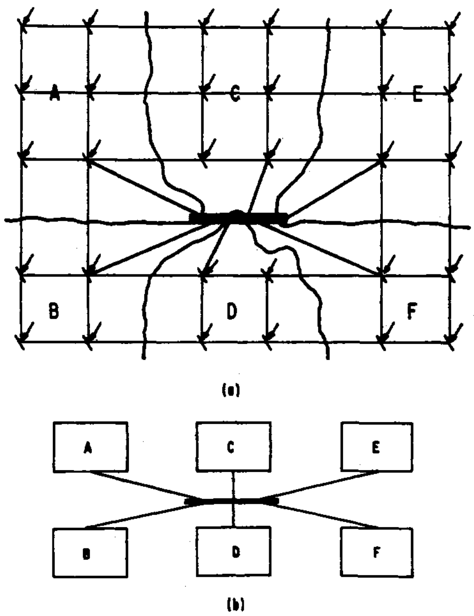


Fig. 5. (a) Network with lines of tear indicated. (b) Subdivisions radially attached—single common bus.

The junction pair voltages for the example network are shown in Fig. 6, and they all take the same form as emanating or spanning the system from the single common bus to all other buses. In general, the junction pairs can span the subdivisions in any manner desired (tree-like manner).

The relationship between the junction pair voltages and the injected currents will first be expressed through a nodal admittance matrix formed for each subdivision which can be

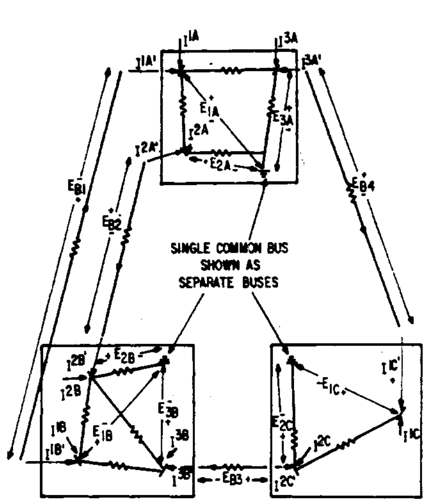
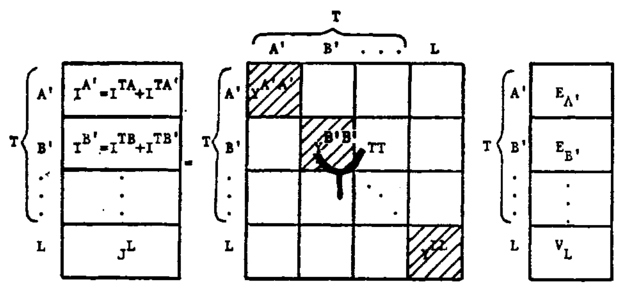


Fig. 6. Torn subdivisions.

directly constructed from the branch admittances. The composite of all the subdivision admittance matrices will be called Y^{TT}.



(4)

Note from (4) that mutual coupling is assumed absent between branches belonging to different subdivisions [3].

Y_{LL} in (4) is a submatrix which represents the intersubdivision branches ($b1$, $b2$, $b3$, $b4$), in the example, and is a branch admittance matrix. Equation (4) will be denoted

$$J^{\tau} = Y^{\tau\tau} V_{\tau}. \tag{5}$$

Voltage vectors $E_{A'}$, $E_{B'}$, $\cdots$, represent junction-pair (open-path) voltages. In the example case

$$E_{A'} = (E_{1A}, E_{2A}, E_{3A})$$
$$E_{B'} = (E_{1B}, E_{2B}, E_{3B})$$
$$E_{C'} = (E_{1C}, E_{2C}). \tag{6}$$

Current vectors $I^{A'}$, $I^{B'}$, $\cdots$, represent junction-pair (open-path) currents, they have been represented as two components: 1) those due to external current sources I^T, and 2) those due to the interconnection of the subdivisions $I^{T'}$

In the example case in Fig. 6

$$I^{TA} = (I^{1A}, I^{2A}, I^{3A})$$
$$I^{TA'} = (I^{1A'}, I^{2A'}, I^{3A'})$$
$$I^{TB} = (I^{1B}, I^{2B}, I^{3B})$$
$$I^{TB'} = (I^{1B'}, I^{2B'}, I^{3B'})$$
$$I^{TC} = (I^{1C}, I^{2C})$$
$$I^{TC'} = (I^{1C'}, I^{2C'}). \tag{7}$$

The inverse of (5) is

$$V_r = Z_{rr}J^r. \tag{8}$$

In expanded form, (8) appears as

$$\begin{array}{c} \\ T\left\{\begin{array}{c} A' \\ B' \\ \vdots \end{array}\right. \\ L \end{array}
\begin{bmatrix} E_{A'} \\ E_{B'} \\ \vdots \\ E_L + e_L \end{bmatrix}
=
\begin{array}{c} \\ T\left\{\begin{array}{c} A' \\ B' \\ \vdots \end{array}\right. \\ L \end{array}
\overset{\overbrace{A' \quad B' \quad \cdots}^{T} \quad L}{\begin{bmatrix} Z_{A'A'} & & & \\ & Z_{B'B'} & & \\ & & Z_{T\cdot T} & \\ & & & Z_{LL} \end{bmatrix}}
\begin{array}{c} \\ T\left\{\begin{array}{c} A' \\ B' \\ \vdots \end{array}\right. \\ L \end{array}
\begin{bmatrix} I^{A'} = I^{TA} + I^{TA'} \\ I^{B'} = I^{TB} + I^{TB'} \\ \vdots \\ J^L \end{bmatrix} \tag{9}$$

Z_{TT} represents the solution of the subdivisions in the absence of their interconnections, since with $I^{T'}$ zero, (9) directly yields the subdivision voltages. Z_{TT} will play a key role throughout this development, and is analogous to the $B_{\alpha\beta}$ of the classic problem in (2).

Equations (4) and (9) represent the model of the network but with the subdivisions not interconnected. An equivalent network can be generated from (9), as shown in Fig. 7, by interpreting the diagonal elements of Z_{TT} as self-impedances through which the open paths flow, and the off-diagonal elements as electromagnetic mutuals between the constructed branches. The mutuals have not been explicitly shown in Fig. 7.

Z_{rr} in (9) need not be computationally obtained from Y^{rr} by inversion. Matrix building techniques can be employed if desired, or other suitable methods. It must be emphasized that Z_{rr} as such is not even required for solving the network problem, as will be further elaborated upon.

In (4) and (9) the subdivisions were given the index T and the intersubdivision the index L. The reason for these indices is readily apparent from Figs. 6 and 7. The procedure of constructing Y^{rr} is equivalent to eliminating the loops in the subdivisions, so that the subdivisions have been reduced to an equivalent tree form. Because the torn subdivisions are radially hinged in this tearing case, the compound tree formed by all the subdivisions is also a tree of the total connected network. These branches therefore bear the index T.

The only remaining loops in the system are formed due to the subdivision interconnecting branches, also called intersubdivision branches; these latter branches therefore bear the index L.

The important point to note is that the subdivision junction pairs span the entire connected network and, when solved, also represent the solution of the total interconnected network. The junction-pair (open-path) voltages are defined as E_T; and are illustrated in Fig. 7:

$$E_T \equiv (E_{A'}, E_{B'}, \cdots). \tag{10}$$

E_T excludes V_L. $E_{A'}$, $E_{B'}$, $\cdots$, are defined in (6).

Junction-pair (open-path) currents are defined as $(I^T + I^{T'})$ illustrated in Fig. 7:

$$I^T \equiv (I^{TA}, I^{TB}, \cdots) \tag{11}$$

$$I^{T'} \equiv (I^{TA'}, I^{TB'} \cdots). \tag{12}$$

I^T and $I^{T'}$ exclude J^L. I^{TA}, I^{TB}, and $I^{TA'}$, $I^{TB'}$ are defined in (7).

Notice that the solution of the subdivisions can readily be obtained by various computational methods from (4) or directly from (9) if $I^{T'}$ in (12) is known. From (9)

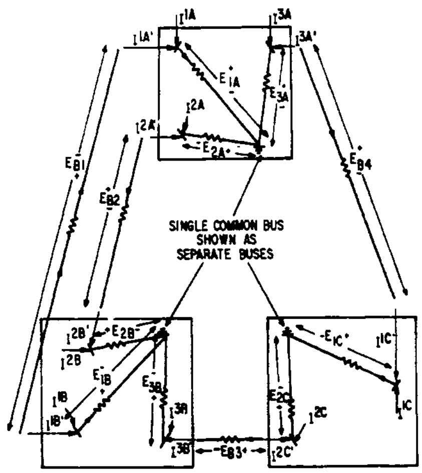


Fig. 7. Equivalent representations of subdivisions, torn subdivisions radially attached (r).

$$E_T = Z_{TT}I^T + Z_{TT}I^{T'}. \tag{13}$$

The equivalent operation to that in (13) can be realized by employing elimination techniques or triangular factorization techniques upon (4). The problem of course is that $I^{T'}$ is unknown.

Currents $I^{T'}$ are similar to tie powers in the classic problem. Their determination will be considered at the next computational level or levels.

A Note on Generality

The selected junction pairs in the subdivisions, as in Fig. 6, are arbitrary and can take any form so long as they span the buses of the subdivision in a tree-like manner (see [3], pp. 251–255). The particular reference frame shown in Fig. 7 is an especially useful one where all equivalent branches emanate from a single common bus.

The excitation considered in the original network can consist of voltage sources in series with each branch, current sources across each branch, and current sources across all junction pairs. All these sources are equivalenced into one set of junction pair currents (see [3], p. 216).

Intersubdivision Level

The problem to be solved is the determination of the currents created due to the cut ($I^{T'}$), and their contribution to the solution of the total interconnected system.

The method to be presented is based upon the network composed of the equivalent branches shown in Fig. 7. These branches can be viewed as forming a new primitive system shown in Fig. 8(a) which, upon interconnection, yields the connected network shown in Fig. 8(b). This new network is identical to that in Fig. 7 but with all the subdivisions interconnected. The new network pictured in Fig. 8(a) is defined by (9) except for notation as shown

$$\begin{bmatrix} V_T = E_T \\ \hline V_L = E_L + e_L \end{bmatrix} = \begin{array}{c} T \\ L \end{array} \overset{\begin{array}{cc} T & L \end{array}}{\left[\begin{array}{c|c} Z_{TT} & 0 \\ \hline 0 & z_{LL} \end{array}\right]} \begin{bmatrix} J^T = I^T + i^T \\ \hline J^L = i^L \end{bmatrix} \tag{14}$$

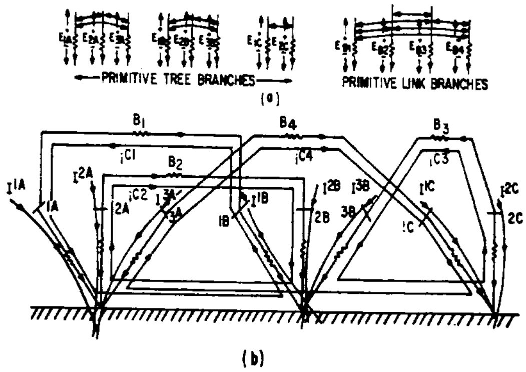


Fig. 8. (a) Newly created primitive network. (b) Network equivalent of original network.

where V_T normally includes voltage sources

$$V_T = E_T + e_T. \tag{15}$$

In this case all voltage sources in the subdivisions have been converted to current sources as indicated earlier, and therefore $e_T = 0$. E_T has been defined in (10), and I^T in (11). $I^{T'}$ in (12) is equal to i^T in (14), and is defined to be due to other than external sources.

$$I^{T'} = i^T. \tag{16}$$

The change from the injected currents $I^{T'}$ to network currents i^T corresponds to the change from torn subdivisions in Fig. 7 to the interconnected network in Fig. 8.

Current sources are not assumed to be present in the intersubdivision branches for convenience of this paper ($I^L = 0$), and therefore J^L in (14) is solely due to i^L. The presence of I^L sources is considered in [3].

The primitive in Fig. 8(a) will next be transformed to the network in Fig. 8(b). A transformation matrix can be established by means of Fig. 8(b) which relates the path currents defined in Fig. 8(b) to those of the branch currents in the primitive.

$$J^r = C^r{}_{.s} J^s \tag{17}$$

where J^r are the currents in the primitive and J^s are the path currents shown in Fig. 8(b). The latter consist of open-path currents I^o and closed path currents i^c. In this particular case (unit tree and unit link), the paths have been chosen in a particular simple manner so that $I^o = I^T$. For a discussion of more general paths [3] should be consulted.

$C^r{}_{.s}$ is a transformation matrix which is illustrated in (18) below. The dot indicates that s is the second index rather than the first, and that the matrix is thus $r \times s$. The transpose of $C^r{}_{.s}$ is $C_s{}^{.r}$, illustrating that it is an $s \times r$ matrix. The dot notation is covered in detail in [3].

Equation (17) in expanded form is

$$\begin{matrix} T \\ L \end{matrix}\begin{bmatrix} J^T = I^T + i^T \\ \hline J^L = i^L \end{bmatrix} = \begin{matrix} T \\ L \end{matrix}\overset{\quad o \qquad\qquad c}{\begin{bmatrix} 1^T{}_{.o} & C^T{}_{.c} \\ \hline & 1^L{}_{.c} \end{bmatrix}} \begin{matrix} o \\ c \end{matrix}\begin{bmatrix} I^o = I^T \\ \hline i^c \end{bmatrix} \tag{18}$$

when transforming (14) by means of $C^r{}_{.s}$, the following equations result

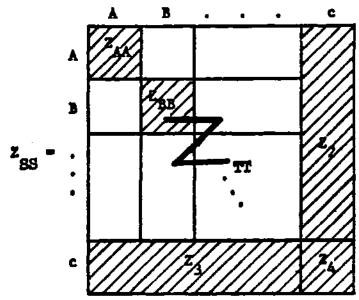
$$V_s = Z_{ss} J^s \tag{19}$$

where $V_s = C_s{}^{.r} V_r$ and $Z_{ss} = C_s{}^{.r} Z_{rr} C^r{}_{.s}$. Equation (19) in expanded form and also writing the expressions of all submatrices is

$$\begin{bmatrix} V_o = E_o \\ E_T \\ \hline V_c = E_c + e_c \end{bmatrix} = \begin{matrix} o \\ c \end{matrix}\overset{\qquad o \qquad\qquad\qquad c}{\begin{bmatrix} Z_1 = 1_o{}^{.T} Z_{TT} 1^T{}_{.o} & Z_2 = 1_o{}^{.T} Z_{TT} C^T{}_{.c} \\ \hline Z_3 = C_c{}^{.T} Z_{TT} 1^T{}_{.o} & Z_4 = C_c{}^{.T} Z_{TT} C^T{}_{.c} + 1_c{}^{.L} Z_{LL} 1^L{}_{.c} \end{bmatrix}} \begin{bmatrix} I^o = I^T \\ \hline i^c \end{bmatrix} \tag{20}$$

where $e_c = 1_c{}^{.L} e_L$ and E_c is usually zero.

Z_{ss} takes the following form



(21)

Z_1 consists of the submatrices of the torn areas $A, B, \cdots$, in (9) as indicated. Z_2, Z_3, and Z_4 are submatrices that reflect the interconnection of the subdivisions, and they can be constructed from Z_1 as shown previously [3], [21].

Note that $Z_1 I^o$ and $Z_2 i^c$ represent $Z_{TT} I^T$ and $Z_{TT} I^{T'}$ in (13), respectively.

With e_c and I^T known in (20), i^c and E_T can be solved from (20)

$$e_c = Z_3 I^o + Z_4 i^c. \tag{22}$$

Solving for i^c from (22)

$$i^c = Z_4^{-1}(e_c - Z_3 I^o). \tag{23}$$

Solving for E_T from (20)

$$E_T = Z_1 I^o + Z_2 i^c. \tag{24}$$

Factorized Solutions of (20)

The solution of (20) can also be obtained without forming Z_2 and Z_3. The factorized solution is given in Table I along with the network models which are required. The derivation of the factorization is presented in detail in [3, table 12.5, p. 243].

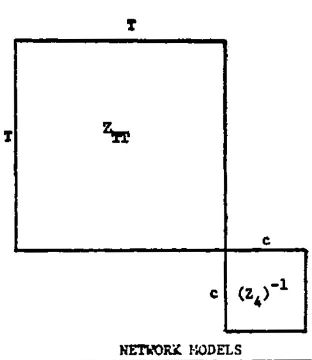
Notice that the algorithm covers the case whose primitive is given in (14). Voltage sources (e_L) are only assumed present in link branches and current sources (I^T) are limited to the tree branches.

Note that step 1 represents the solution of the subdivisions. Steps 2 through 4 compute the net voltages in the intersubdivision network which is the equivalent of the original network pictured for the example case in Fig. 8(b). Step 5 computes the closed-path currents and the resulting injected currents i^T. Since $i^T = I^{T'}$, from (16), step 7 represents $Z_{TT} I^{T'}$ in (13) which is the contribution due to the interconnection.

Matrix $C^T{}_{.c}$ is obviously important in the execution of the piecewise algorithm in Table I. It is not explicitly required.

TABLE I
FACTORIZED SOLUTION—TORN SUBDIVISION RADIALLY ATTACHED

1. $V_T^{(o)} = Z_{TT} I^T$
2. $e_c = I_c^{\cdot L} e_L$
3. $V_c^{(o)} = C_c^{\cdot T} V_T^{(o)}$
4. $e_c' = e_c - V_c^{(o)}$
5. $i^c = (Z_4)^{-1} e_c'$
6. $i^T = C^T_{\cdot c} i^c$
7. $V_T^{(1)} = Z_{TT} i^T$
8. $V_T = V_T^{(o)} + V_T^{(1)}$



For the example case in Fig. 8(b), where all subdivisions emanate from a common reference, $C^T_{\cdot c}$ simply assigns signs to currents i^c as seen from the example case in Fig. 8(b).

$$\begin{array}{c} 1A\\2A\\3A\\1B\\2B\\3B\\1C\\2C \end{array}\begin{bmatrix} I^{1A'}\\ I^{2A'}\\ I^{3A'}\\ I^{1B'}\\ I^{2B'}\\ I^{3B'}\\ I^{1C'}\\ I^{2C'} \end{bmatrix} = \begin{array}{c} 1A\\2A\\3A\\1B\\2B\\3B\\1C\\2C \end{array}\overset{\begin{array}{cccc} c1 & c2 & c3 & c4 \end{array}}{\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & & -1\\ -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\\ & & -1 & \end{bmatrix}} \begin{array}{c} c1\\c2\\c3\\c4 \end{array}\begin{bmatrix} i^{c1}\\ i^{c2}\\ i^{c3}\\ i^{c4} \end{bmatrix}$$

In a more general case, a combination of closed-paths currents comprises each component of $I^{T'}$ [3].

Step 3, as proven elsewhere [3], represents the negative of the voltages across the torn subdivisions labelled $E_L^{(o)}$

$$E_L^{(o)} = - C_c^{\cdot T} V_T^{(o)}. \tag{25}$$

Equation (25) is general and thus step 4 in Table I can be written

$$e_{c'} = e_c + E_L^{(o)}. \tag{26}$$

The algorithm in Table I can be simplified to the following steps

1) Obtain solution of torn subdivisions excluding tie currents to other subdivisions ($V_T^{(o)} = Z_{TT} I^T$).
2) $e_c = I_c^{\cdot L} e_L$.
3) Compute voltages across torn subdivisions ($E_L^{(o)}$) given intersubdivision (CUT) branch sign convention.
4) $$e_{c'} = e_c + E_L^{(o)}.$$
5) Compute closed-path currents ($i^c = Z_4^{-1} e_{c'}$).
6) Convert closed-path currents (i^c) to injected tie currents. I^{ties} ($I^{T'}$) by assigning signs.
7) Obtain voltage contributions in subdivisions due to tie currents $I^{T'}$. ($V_T^{(1)} = Z_{TT} I^{T'}$).
8) Total voltage solution is the sum of the voltages obtained in steps 1) and 7). ($V_T = V_T^{(o)} + V_T^{(1)}$).

Simplified factorized solution—torn subdivisions radially attached; steps 3) and 6) assume subdivisions of the form shown in Fig. 8(b) and the paths as shown.

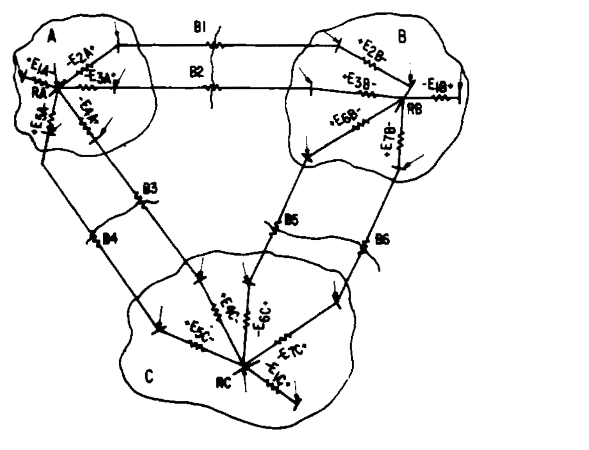


Fig. 9. Equivalent representation of subdivisions, torn subdivisions not attached (r). (RA, RB, RC represent reference buses.)

Multilevel Theory

The preceding tearing algorithms have considered the interconnections in a power system at two levels: 1) the subdivision level for recognizing the interconnections within the subdivisions (index 0) and 2) the intersubdivision level for recognizing the interconnections of the subdivisions themselves (index 1). The previous work can be generalized from two levels to any number of computing levels (0, 1, 2, 3 $\cdots$) [3], [22].

The contours that were utilized above were unit tree and unit link. The reader is referred to [3] where algorithms for the case with a more general form of the primitive (coupling between tree and link are considered) is given, and where more general excitations and more general contours are assumed.

TORN SUBDIVISIONS NOT ATTACHED

Let us now consider the network in Fig. 3, but where the torn subdivisions are not attached.

An example case is that in Fig. 6 with the exception that areas A, B, C would not be grounded; with branches ($B1$, $B2$, $B3$, $B4$) removed, the subdivisions are not attached.

We again start the piecewise analysis with the subdivision solutions, which will not be repeated here since the material presented for the radially attached case is equally valid for the not attached case. Equations (5) and (8), or in expanded form, (4), (9), and (14) apply. Equation (14) will be restated

$$\begin{bmatrix} V_T = E_T \\ \hline V_L = E_L + e_L \end{bmatrix} = \begin{array}{c} T \\ L \end{array}\overset{\begin{array}{cc} T & L \end{array}}{\left[\begin{array}{c|c} Z_{TT} & 0 \\ \hline 0 & Z_{LL} \end{array}\right]} \begin{bmatrix} J^T = I^T + i^T \\ \hline J^L = i^L \end{bmatrix} \tag{14}$$

T—refer to the subdivision branches which in this case are not connected; L—refer to the intersubdivision branches. Note that Z_{TT} in (14) is block diagonal as shown in (9).

The major exception of this case, vis a vis the previous, is that the junction pairs (T) of the subdivisions no longer are sufficient to describe the total interconnected network.

We will illustrate the theoretical development by the example in Fig. 9. It represents the equivalent branches of the subdivisions similar to Fig. 7, with the exception that the torn subdivisions are no longer radially attached when branches ($B1 \cdots B6$) are removed. Note that two other junction pairs

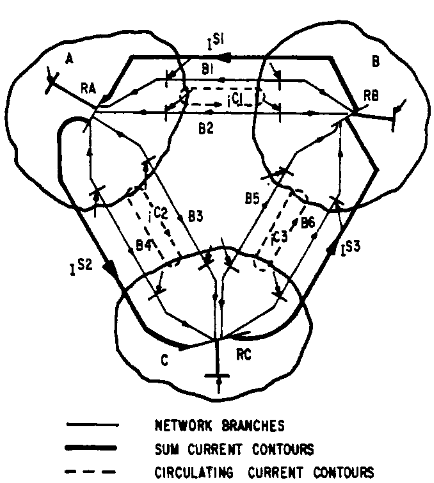


Fig. 10. Contours drawn upon (interconnecting) the system.

are required to span the total network. In the example in Fig. 9, the subdivisions all have the same form as emanating from a common subdivision bus by having chosen junction pairs of that form within the subdivisions. The material equally applies to the case where junction pairs have a different form.

In the subsection to follow, transformations will be presented that will radially attach the subdivisions [2]. The ties between the torn subdivisions ($B1 \cdots B6$) can be looked upon as representing a primitive system of the interchange network, which in the example of Fig. 9 consists of six branches. The topology of the interchange primitive has so far not been recognized but will be considered in the consecutive transformations introduced here.

The first transformation recognizes the connection between adjacent subdivisions. This is accomplished by defining a new set of currents (J^1) comprised of sum and circulating currents illustrated in Fig. 10. The number of sum and circulating variables ($S+CM$) exactly equals the number of inter-subdivision branches (L)

$$L = S + CM. \tag{27}$$

The number of sums equals the number of adjacent interconnections. The transformation equations for transforming the contour currents J^1 in Fig. 10 to primitive currents J^r can be written from Fig. 10, and for a general network takes the following form

$$J^r = C^r{}_{.1} J^1 \tag{28a}$$

or

$$\begin{array}{c} \\ \left[\begin{array}{c} J^T \\ \hline J^L \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{ccc} o & S & CM \end{array} \\ \begin{array}{c} T \\ L \end{array} & \left[\begin{array}{c|c|c} 1^T{}_{.o} & C^T{}_{.S} & C^T{}_{.CM} \\ \hline & C^L{}_{.S} & C^L{}_{.CM} \end{array}\right] \end{array} \left[\begin{array}{c} I^o = I^T \\ \hline I^S \\ \hline i^{CM} \end{array}\right] \tag{28b}$$

For a detailed example, [2] should be consulted. Note that S stands for sum; CM stands for circulating; o refers to open path which are unit tree [3] here which means that each open path traces out one and only one tree branch.

Note that the equations of the Lth row in (28) as well as in later steps are not required if the system is torn at the boundary buses with the area tie buses included in the tree designation (T) [23]. The inclusion of the L rows allows tearing through branches as $B1$–$B6$ in Fig. 10 which are not a part of the area subdivisions, and represents a somewhat more general tearing case.

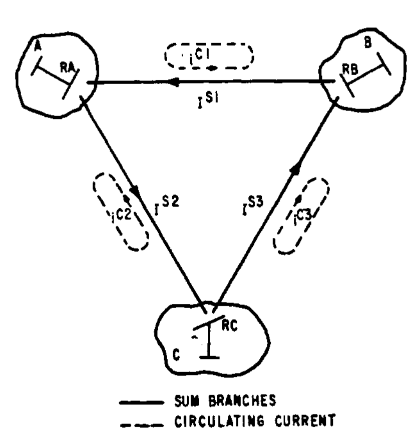


Fig. 11. Primitive interchange (sum) network 1.

After the transformation of voltage $V_r (V_1 = C_1{}^{\cdot r} V_r)$ and $Z_{rr} (Z_{11} = C_1{}^{\cdot r} Z_{rr} C^r{}_{.1})$ and or admittances (Y^{rr}) the following equations result

$$V_1 = Z_{11} J^1. \tag{29a}$$

Expanding (29a)

$$\left[\begin{array}{c} V_o = E_T \\ \hline V_S = E_S + e_S \\ \hline V_{CM} = E_{CM} + e_{CM} \end{array}\right] = \begin{array}{cc} & \begin{array}{ccc} o & S & CM \end{array} \\ \begin{array}{c} o \\ S \\ CM \end{array} & \left[\begin{array}{c|c|c} Z_1 = Z_{TT} & Z_{2'} & Z_{2''} \\ \hline Z_{3'} & Z_5 & Z_6 \\ \hline Z_{3''} & Z_7 & Z_8 \end{array}\right] \end{array} \left[\begin{array}{c} I^o = I^T \\ \hline I^S \\ \hline i^{CM} \end{array}\right] \tag{29b}$$

E_{CM} is zero from Kirchhoff's law, e_S and e_{CM} are active voltage sources, which due to the absence of voltage sources in the tree branches, are ($e_S = C_S{}^{\cdot L} e_L$, $e_{CM} = C_{CM}{}^{\cdot L} e_L$) [2].

E_S are junction pair voltages which represent the extremities of the open paths. For the example case in Fig. 10 and the current directions chosen, row 1 of E_S is ($E_{RB} - E_{RA}$), row 2 is ($E_{RA} - E_{RC}$), and row 3 is ($E_{RC} - E_{RB}$).

Equation (29b), in the case when currents (I^S) and (I^T) are known, can be solved directly for E_T, E_S, and i^{CM}; E_{CM} is zero, and e_S and e_{CM} follow from e_L. Equation (29b) can likewise be solved for the case where the reference voltages E_{RA}, E_{RB}, E_{RC}, and thus E_S are known instead of I^S. Since I^S and the reference voltages are usually not known, except in the two-area case, additional transformations are required to produce equations with variables which are known. The variables desired are the area net interchange currents which are area-excess currents; they will be designated by the symbol I^{Ek}, where E stands for subdivision excess out of each area, and is generally known.

Before the transformation, a network representing (29b) has to be constructed. As in the case of preceding equations, (29b) can be interpreted in network form as shown in Fig. 11. Z_{TT} nor Z_8 of (29b) have been shown in detail in Fig. 11 but only the network equivalents of the Z_5 submatrix. As indicated previously, the equivalent network branches lie along the junction pairs (open paths) selected; thus Z_5 for the exam-

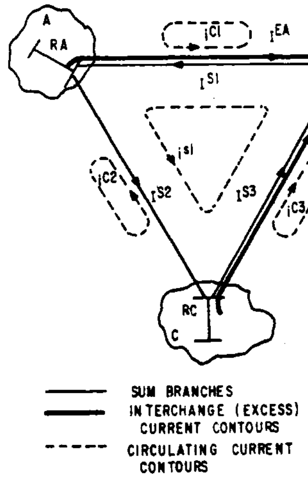


Fig. 12. Contours drawn upon (interconnecting) the interchange network.

ple case consists of three branches as indicated, and form a closed path. The sum branches can be interpreted to represent a primitive interchange network, in that the interconnection of the sum branches themselves has not been recognized. A transformation matrix can be defined in which the tree (T) and the (CM) variables remain unchanged, but where the sum branches are transformed to contour variables. The open-path variables of the interchange network are the I^{Ek} currents defined above and are generally known; they are shown in Fig. 12. The closed-path currents of the interchange network are called sneak currents (i^{sN}), and they circulate in the meshes formed by the sums. In the particular case shown in Fig. 11, there are three sum branches with one mesh or closed path formed. The primitive interchange network can be considered to be an ordinary network with the areas representing nodes. The number of sum branches are thus equal to the number of interchange plus the number of sneak flows:

$$\text{number of sums } (S) = \text{number of interchanges } (Ek) + \text{number of sneaks } (sN). \quad (30)$$

Since the interchanges span the areas of the system

$$\text{number of interchanges } (Ek) = \text{number of areas} - 1. \quad (31)$$

In the example case in Fig. 11, two independent interchanges and one sneak path exist, as shown in Fig. 12.

The transformation equations for interconnecting the primitive interchange network can be obtained by expressing currents J^1 in (29) in terms of new currents J^2, and can be written from Fig. 12; for a general network, the transformation equations take the following form

$$J^1 = C^1{}_{.2}J^2. \quad (32a)$$

Equation (32a) in expanded form is as follows

$$\begin{array}{c}\\ \left[\begin{array}{c} I^o = I^T \\ I^S \\ i^{CM} \end{array}\right]\end{array} = \begin{array}{c} \quad\ \ o \quad\ Ek \quad\ sN \quad\ CM \\ \begin{array}{c} o \\ S \\ CM \end{array}\left[\begin{array}{cccc} 1^o_{.o} & & & \\ & C^S_{.Ek} & C^S_{.sN} & \\ & & & 1^{CM}_{.CM} \end{array}\right]\end{array} \left[\begin{array}{c} I^o = I^T \\ I^{Ek} \\ i^{sN} \\ i^{CM} \end{array}\right] \quad (32b)$$

We can, alternatively, directly transform J^2 in (32a) to J^r in (28) by substituting (32a) into (28a)

$$J^r = C^r{}_{.1}C^1{}_{.2}J^2 \quad (33)$$

the single matrix $C^r{}_{.2}$ is defined to be

$$C^r{}_{.2} = C^r{}_{.1}C^1{}_{.2}. \quad (34)$$

$C^r{}_{.2}$ transforms J^2 to J^r and takes the following form

$$J^r = C^r{}_{.2}J^2. \quad (35)$$

$C^r{}_{.2}$ in expanded form is

$$C^r{}_{.2} = \begin{array}{c} \quad\ o \quad\ Ek \quad\ cn \\ \begin{array}{c} T \\ L \end{array}\left[\begin{array}{ccc} 1^T_{.o} & C^T_{.Ek} & C^T_{.cn} \\ & C^L_{.Ek} & C^L_{.cn} \end{array}\right]\end{array} \quad (36)$$

where $cn = sN + CM$. Numerical examples of $C^1{}_{.2}$ and of $C^r{}_{.2}$ appear in [2].

After transforming (8) by means of $C^r{}_{.2}$, or transforming (29) through $C^1{}_{.2}$, the following equations result

$$V_2 = Z_{22}J^2. \quad (37)$$

V_2 transforms from V_1 or from V_r, and Z_{22} likewise transforms from Z_{11} or Z_{rr}.

Expanding (37) in terms of components of Z_{rr} and $C^r{}_{.2}$ we obtain a set of equations analogous to (20)

$$\left[\begin{array}{c} V_o - E_o - E_T \\ V_{Ek} - E_{Ek} + e_{Ek} \\ V_{cn} - E_{cn} + e_{cn} \end{array}\right] = \begin{array}{c} \overbrace{\quad o \qquad\qquad Ek \quad}^{o'} \qquad\quad cn \\ \begin{array}{c} o \\ Ek \\ cn \end{array}\left[\begin{array}{ccc} Z_1 = 1_o^{.T}Z_{TT}1^T_{.o} & Z_2' = 1_o^{.T}Z_{TT}C^T_{.Ek} & Z_2'' = 1_o^{.T}Z_{TT}C^T_{.cn} \\ Z_3' = C_{Ek}^{.T}Z_{TT}1^T_{.o} & Z_5 = C_{Ek}^{.T}Z_{TT}C^T_{.Ek} + C_{Ek}^{.L}Z_{LL}C^L_{.Ek} & Z_6 = C_{Ek}^{.T}Z_{TT}C^T_{.cn} + C_{Ek}^{.L}Z_{LL}C^L_{.cn} \\ Z_3'' = C_{cn}^{.T}Z_{TT}1^T_{.o} & Z_7 = C_{cn}^{.T}Z_{TT}C^T_{.Ek} + C_{cn}^{.L}Z_{LL}C^L_{.Ek} & Z_8 = C_{cn}^{.T}Z_{TT}C^T_{.cn} + C_{cn}^{.L}Z_{LL}C^L_{.cn} \end{array}\right]\end{array} \left[\begin{array}{c} I^o = I^T \\ I^{Ek} \\ i^{cn} \end{array}\right] \quad (38)$$

where

$$(e_{Ek} = C_{Ek}{}^{.L}e_L, \quad \text{and} \quad e_{cn} = C_{cn}{}^{.L}e_L).$$

E_{cn} is usually zero. (38) is interpreted in network form in Fig. 13. E_{Ek} of (38) in the example case consists of differences of potential of the area references: from Fig. 13, row 1 of E_{Ek} is $(E_{RA} - E_{RB})$ and row 2 is $(E_{RC} - E_{RB})$. We notice that the torn subdivisions which are not attached in Fig. 9 are again radially attached and shown in Fig. 13. Thus the transformations reintroduced the necessary fictitious branches and thereby satisfying the requirement that the "open paths of the total network be radially attached." Other fictitious branches or devices can be used also to satisfy the "radially attached" criterion, and the above (sum and interchange) variables are not the only ones that can be employed.

Note the o' designation in (38). It indicates the open paths

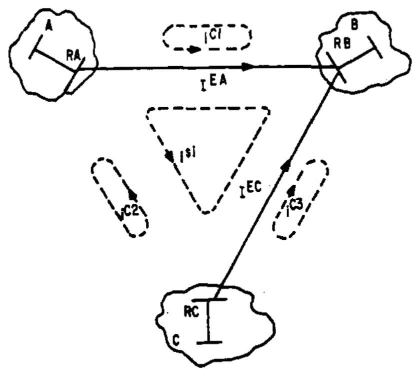


Fig. 13. Interconnected interchange network 2.

in Fig. 13, which include those of the interchange network. The intersubdivision network consists of both the circulating and the sneak paths indicated by the cn axes.

Notice, also, that despite the transformations, the block diagonal form of Z_{TT} has been maintained. Sparsity has thus been kept intact as represented by the solution of the torn subdivisions.

Modifications in the above transformation equations can be made. In [23], for example, the above transformations have been made with no Lth axes present in (14), (28), and (36) due to the fact that the tearing in [23] was conducted at company corporate boundaries with no link branches present. The present treatment allows tearing through branches as well as buses as mentioned previously. Subdivision branches traced out by contours in the intersubdivision interconnection process and present in the primitive (Z_{TT}) in (9) were not retained in succeeding reference frames 1 and 2 (Z_{11} and Z_{22}) in the work in [23] because no applied currents were present at the extremeties of those particular branches; they have been retained throughout the transformation here to allow currents to be present at the extremities of these branches as shown in Fig. 9.

Equation (38) can be solved for E_T, E_{Bk}, and i^{cn} with applied currents I^T and I^B known, and all voltage sources e_L known. Interchange currents I^{Ek} can be calculated from I^T and I^B; this is the common case. Alternatively, (38) can be solved for the case where the reference voltages, and thus E_{Bk}, are known instead of I^B and I^{Ek}.

A Note of Generality

Note that $1^T{}_{.o}$ appears in all transformations in this text, which indicates that the contours in the primitive (unit tree) are maintained in the interconnected system; this is not a requirement. Contours in the interconnected system can be different from those in the primitive (14), in which case a $C^T{}_{.o}$ is required in place of the $1^T{}_{.o}$ as shown in [3]; the piecewise algorithms in Tables I and II, in that case, would be preceded by a transformation which transforms the open-path currents to primitive currents by means of the $C^T{}_{.o}$.

Factorized Solution of (38)

The solution of (38) can be obtained without requiring the Z_2 and Z_3 submatrices. The factorized solution is given in Table II along with the network models which are required. The derivation is presented in detail in [2] and will not be reproduced here.

TABLE II

FACTORIZED SOLUTION—TORN SUBDIVISION NOT ATTACHED

1. $V_T^{(o)} = Z_{TT} I^T$
2. $e_{cn} = C_{cn}^{.L} e_L$
3. $V_c^{(o)} = C_{cn}^{.T} V_T^{(o)}$
4. $e'_c = e_{cn} - V_c^{(o)} - Z_7 I^{Ek}$
5. $i^c = (Z_8)^{-1} e'_c$
6. $i^T = C^T{}_{.Ek} I^{Ek} + C^T{}_{.cn} i^{cn}$
7. $V_T^{(1)} = Z_{TT} i^T$
8. $V_T = V_T^{(o)} + V_T^{(1)}$

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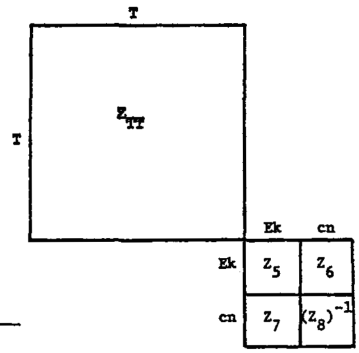
9. $V_{Ek} = C_{Ek}^{.T} V_T^{(o)} + Z_5 I^{Ek} + Z_6 i^{cn}$



Network Models

The C submatrices in Table II are those in (36). Numerical examples of the C submatrices appear in [2]. Z_{TT} is as shown in (9), and represents the subdivision solution, and the (Z_5–Z_8) cluster is that in (38).

Step 1 in Table II represents the solution of the subdivisions with no contributions of other subdivisions considered. Steps 2 through 4 compute the net voltages in the intersubdivision network represented by the Z_8 submatrix. Notice the contribution of the interchange network to $e_{c'}$ in step 4. Step 5 computes the closed-path currents i^{cn} which consists of i^{CM} and i^{sN} currents, pictured in the example case in Fig. 13. Step 6 computes the injected currents, which in this case are comprised of both interchange and circulating currents. Step 7 calculates the voltages contributions to each subdivision due to its interconnection with other subdivisions; and step 8 represents the total solution. Step 9 is not required but may be calculated for other uses [2].

It should be emphasized that the steps in the factorized solutions in Tables I and II for practical applications are carried out implicitly, since elimination or triangular factorization techniques, and sparsity programming are employed.

The important point to note is that tearing with "subdivisions not attached" only represents a minor modification of the "radially attached case" as seen by comparing Table II with Table I. Note also that the cluster Z_5–Z_8 in Table II is of the same order as that of Z_4 in Table I.

In the preceding procedure the stepwise process recognizes all the interconnections at one time, i.e., by means of (1) level. The interconnections can also be recognized by means of a multiplicity (n) of levels as shown in [3] and [22]. The steps themselves can be lengthened or combined if desired, depending upon the application under consideration.

Modifications in the algorithms can readily be made. The steps can be dramatically shortened in the piecewise algorithms by using a tie model instead of the Z_5–Z_8 cluster; a tie model called an "interarea matrix" can be generated as described in [23] which explicitly expresses all the tie currents created in the tearing operation as a function of all the applied currents (I^o) and the interchange currents I^{Ek}. The solution algorithm to produce voltages similar to those in Table II but

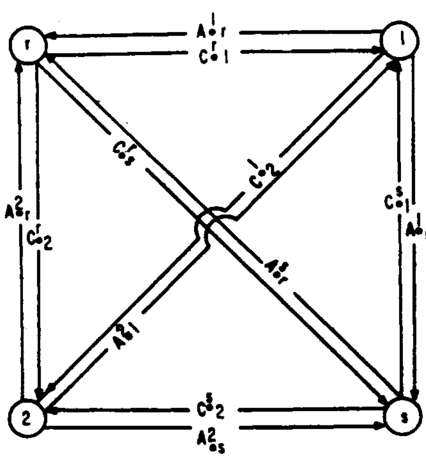


Fig. 14. Interrelationships between reference frames.

using the interarea matrix would proceed in two steps: 1) all tie currents are calculated directly from the interarea current model and the currents given, and 2) each subdivision is solved from the applied currents and the tie currents just calculated. The solution represents the entire system solution, since the isolated or torn subdivisions are solved simultaneously; given all the currents applied, including those from the neighboring subdivisions.

The solution to the classic loss problem follows as outlined in 1) and 2) above with the exception that powers are used instead of currents. In 1) tie powers (P^{TA}, P^{TB}, $\cdots$) are calculated from the interarea power model [23] and the powers given, and 2), (3) can be solved for the total losses of the system.

Relationships Between Reference Frames

Four reference frames have so far been employed for the total interconnected network: the primitive reference frame (r), torn subdivisions radially attached (s), torn subdivisions not attached (1), (2).

Transformations exist, as derived in [2], for passing from any of the four reference frames to any other. One of the practical results that are realized is that the (Z_5–Z_8) cluster in Table II can be formed from the Z_4 of Table I. Since Z_4 is readily constructed [3], the (Z_5–Z_8) cluster can also be readily formed.

The four reference frames are shown as circles in Fig. 14 as well as the transformations between the reference frames which serve as roads between them. Note that the following four transformation matrices have been previously defined in the text: $C^r{}_{.s}$ in (17), whose inverse is $A^s{}_{.r}$; $C^r{}_{.1}$ in (28), whose inverse is $A^1{}_{.r}$; $C^1{}_{.2}$ in (32), whose inverse is $A^2{}_{.1}$; $C^r{}_{.2}$ in (35), whose inverse is $A^2{}_{.r}$. Equation (34) demonstrates the use of Fig. 14.

$$C^r{}_{.2} = C^r{}_{.1}C^1{}_{.2}. \tag{34}$$

We can determine $C^r{}_{.1}$ in (28) and $C^r{}_{.2}$ in (36) from $C^r{}_{.s}$ or vice versa by means of Fig. 14 as follows [2]

$$C^r{}_{.1} = C^r{}_{.s}C^s{}_{.1} \tag{39}$$

and

$$C^r{}_{.2} = C^r{}_{.s}C^s{}_{.2} \tag{40}$$

or $C^r{}_{.s}$ can likewise be determined from $C^r{}_{.1}$, or from $C^r{}_{.2}$.

The author [2] derived the above relationships, and others, and shows that the form of $C^s{}_{.2}$ is

$$C^s{}_{.2} = \begin{array}{c|c|c|c|} & o & Ek & cn \\ \hline o & 1^o{}_{.o} & & \\ \hline c & & C^c{}_{.Ek} & C^c{}_{.cn} \\ \hline \end{array} \tag{41}$$

Z_{22} in (38) can be expressed in terms of Z_{ss} in (21) as follows

$$Z_{22} = \begin{array}{c|c|c|c|} & o & Ek & cn \\ \hline o & Z_1 = 1^{\cdot T}_o Z_{TT} 1^T{}_{.o} & Z_2' = Z_2 C^c{}_{.Ek} & Z_2'' = Z_2 C^c{}_{.cn} \\ \hline Ek & Z_3' = C^{\cdot c}_{Ek} Z_3 & Z_5 = C^{\cdot c}_{Ek} Z_4 C^c{}_{.Ek} & Z_6 = C^{\cdot c}_{Ek} Z_4 C^c{}_{.cn} \\ \hline cn & Z_3'' = C^{\cdot c}_{cn} Z_3 & Z_7 = C^{\cdot c}_{cn} Z_4 C^c{}_{.Ek} & Z_8 = C^{\cdot c}_{cn} Z_4 C^c{}_{.cn} \\ \hline \end{array} \tag{42}$$

where Z_1–Z_4 are the submatrices of (38) and $C^c{}_{.Ek}$, $C^c{}_{.cn}$ and their transposes are those of $C^s{}_{.2}$ in (41).

A major result represented by (42) is that the (Z_5–Z_8) cluster has been expressed in terms of Z_4.

The solution procedure with "subdivisions not attached," can proceed from the subdivision solutions (Z_{TT}) and Z_4, which are the network models for the "radially attached case" in Table I. The cluster (Z_5–Z_8) in Table II for the "not attached" case is next calculated, and the network models for the stepwise procedure in Table II thus obtained. Note that the (Z_5–Z_8) cluster can be built by building rules [3], [21] from Z_4 rather than by carrying out the explicit matrix operations above.

Applications

Applications of the above theory to power systems this author has undertaken will be indicated here and in the references and encompass steady-state power system problems (load flow and fault), transient stability problems, as well as control and dispatch problems. The fundamentals of these topics are covered separately in this issue, and will, therefore, not be described here.

Application of diakoptics to a load flow (Z matrix) was first described in a G. E. Data Folder [24] in 1964, which contains the summary of the linear piecewise algorithm, the building rules of all necessary models, and the presentation of the piecewise Z matrix load flow algorithm itself; the implementation of [24] into program form was undertaken by this author cooperatively with engineers from Commonwealth Edison, a power company located in the Chicago area; the methods and results appear in [25] and [26]. Application of diakoptics to another load flow (Newton–Raphson) and to stability are contained in [27].

Algorithms for three-phase, single-phase, etc., fault programs have also been generated [28], and a three-phase tearing program was just recently written by a utility on a cooperative basis with this author.

The piecewise algorithms can either be implemented in single or in multicomputer configurations. A multicomputer

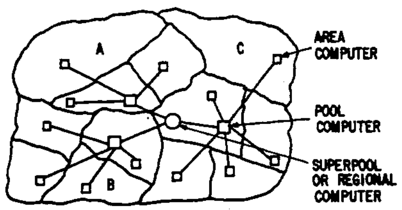


Fig. 15. Diagrammatic example of a multicomputer configuration formed by a three-pool region (*A*, *B*, and *C*), with area subdivisions within each pool.

configuration consists of a hierarchy of computers which are linked to each other by communication channels [29], [30]. The computers can either be physically next to each other, thus forming a cluster of computers, or they can be miles apart.

The first application of diakoptics executed in a multicomputer configuration is the automatic generation control (AGC) of power pools and super pools presently being implemented in the United States. In that case, the multicomputer configuration consists of the dispatch-security computers of the member companies of the pool, and a central or pool computer at the next level and so on as shown in Fig. 15 for a super pool. The application of diakoptics to power system generation control and to power system dispatch was first described in [29] with detailed modeling and methods of generation control of power pools developed in [30]–[34]. Power system dispatch algorithms for power pools contained in [29] were implemented into program form by Niagara Mohawk personnel, a power company located in upstate New York, in a cooperative undertaking with this author; the methods and results appear in [35] and [36]. Piecewise power system load flow and power system stability can also be implemented in multicomputer configurations by means of either a cluster of computers or in available dispatch-security computers as shown in Fig. 15, and tests have been conducted by this author proving feasibility [37].

Conclusions

This paper has provided an overview of the theory of diakoptics and a summary of the applications that have been carried out to date in the power industry.

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