

A Quasi-Convex Optimization Approach to Parameterized Model Order Reduction

Kin Cheong Sou^{*}
kcsou@mit.edu

Alexandre Megretski
ameg@mit.edu

Luca Daniel
luca@mit.edu

ABSTRACT

In this paper an optimization based model order reduction (MOR) framework is proposed. The method involves setting up a quasi-convex program that explicitly minimizes a relaxation of the optimal \mathcal{H}_∞ norm MOR problem. The method generates guaranteed stable and passive reduced models and it is very flexible in imposing additional constraints. The proposed optimization approach is also extended to parameterized model reduction problem (PMOR). The proposed method is compared to existing moment matching and optimization based MOR methods in several examples. A PMOR model for a large RF inductor is also constructed.

Categories and Subject Descriptors: J.6 [COMPUTER-AIDED ENGINEERING]: Computer-aided design (CAD).

General Terms: Algorithm, Theory, Design.

Keywords: parameterized model order reduction, quasi-convex optimization, ellipsoid algorithm, RF inductor.

1. INTRODUCTION

Developing Parameterized Model Order Reduction (PMOR) algorithms would allow digital, mixed signal and RF analog designers to promptly instantiate field solver accurate small models for their parasitic dominated components (interconnect, RF inductors, MEM resonators etc.). The few existing PMOR techniques are based either on statistical performance analysis [15, 9, 20] or on moment matching [19, 23, 7, 13]. Some non-parameterized model order reduction or identification techniques based on an optimization approach are present in literature. [14] and [3] identify systems from sampled data by essentially solving the Yule-Walker equation from a linear least squares problem. However, these methods might not be satisfactory since the objective of their minimization is not the norm of the error system, but rather the same quantity multiplied by the denominator of the reduced model. [8] and [4] directly formulate the model reduction problem as a rational fit minimizing the \mathcal{H}_2 norm of the error system and therefore they solve a non-linear least squares problem, which is not convex. To address the

^{*}K.C. Sou, A. Megretski and L. Daniel are with the department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge MA, 02139, USA. website: <http://www.rle.mit.edu/cpg/>

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DAC 2005, June 13–17, 2005, Anaheim, California, USA.
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problem, these papers propose solving linear least squares iteratively, but it is not clear whether the procedure will converge and whether they can handle additional constraints such as positive realness. In order to reduce positive real systems, [5] proposes using the KYP Lemma and they show that the reduction problem can be cast into a semidefinite program, if the poles of the reduced models are given a priori. [6] uses a different positive realness check from [1] which amounts to a set of scalar inequalities evaluated at some frequency points. [6] then suggests an iterative scheme that minimizes the \mathcal{H}_2 norm of the error system for the frequency points given in the previous iteration. However, this scheme does not necessarily generate optimal reduced models, since in order to do that, both the system model and the frequency points should be considered as decision variables. In short, the available methods lack one or more of the following desirable properties: rational fit, convexity, optimality or flexibility to impose constraints.

In principle, the method proposed in this paper is a rational approximation, but with the following distinctions. Instead of solving the model reduction directly, the proposed methodology solves a relaxation of it. Additionally, the objective function to be minimized is not \mathcal{H}_2 norm but \mathcal{H}_∞ norm. As it turns out, the resultant optimization problem, as described in Section 3, is equivalent to a quasi-convex program i.e. optimization of a quasi-convex function (all sub-level sets are convex sets) over a convex set. This property implies the following: 1) there exists a unique optimal solution to the problem; 2) there exist polynomial time algorithms for solving it. Also, since the proposed method involves only a single optimization problem, it is near optimal with respect to the objective function used (\mathcal{H}_∞ norm of error). In addition to the mentioned benefits, it will be demonstrated in the paper that some commonly encountered constraints can be added to the proposed optimization setup *without* significantly increasing the complexity of the problem. Among these constraints are stability, positive realness, bounded realness, quality factor error minimization. Also, the optimization setup can be modified to generate an optimal parameterized reduced model that is guaranteed stable for the range of parameters of interest.

The rest of the paper is organized as follows: Section 2 provides some background. Section 3 describes the proposed relaxation framework and explains why it is quasi-convex after reparameterization. A procedure for constructing the reduced model is also described. Section 4 demonstrates how to modify the optimization setup to incorporate various desirable constraints. Section 5 focuses on the extension of the optimization setup to the case of parameterized model order reduction. In Section 6 summary of the proposed algorithms are given. In Section 7 some applications examples are shown to evaluate the practical value of the proposed method in terms of accuracy and complexity.

2. BACKGROUND

Given a continuous-time (CT) system with transfer matrix $H(s)$,

a standard technique in the linear system community for reducing it is to first apply a Tustin transform [11] $s = \lambda(z-1)/(z+1)$, to construct an equivalent discrete-time (DT) system, then to reduce the DT system, and finally to convert back to CT. The frequency responses of the CT and DT systems are frequency axis scaled versions of each other and no aliasing occurs because of the Tustin transform. In addition, the effect of frequency warping cancels with the two C/D and D/C conversions. Therefore, the overall reduction quality depends only on the approximation quality of the DT reduction.

One of the desirable model reduction problems is the \mathcal{H}_∞ norm optimization: given a stable transfer function $H(z)$ (of order possibly infinite) and a finite order m , construct a stable rational transfer function $\hat{H}(z) = \frac{p(z)}{q(z)}$ such that order of $\hat{H}(z) \leq m$ and the error $\|H(z) - \hat{H}(z)\|_\infty$ is minimized.

$$\begin{aligned} & \underset{p,q}{\text{minimize}} && \|H(z) - \frac{p(z)}{q(z)}\|_\infty \\ & \text{subject to} && \deg(q) = m, \quad \deg(p) \leq m, \\ & && q(z) \text{ is a Schur polynomial.} \end{aligned} \quad (1)$$

Unfortunately program (1) is not convex and it is not known whether it is \mathcal{NP} complete or not. In other words, existence of an efficient algorithm for solving program (1) is still an open question.

3. RELAXATION SCHEME SETUP

3.1 Relaxation of \mathcal{H}_∞ norm optimization

Motivated by the Hankel optimal model reduction, a relaxation to the optimal \mathcal{H}_∞ norm model reduction is proposed as follows.

$$\begin{aligned} & \underset{p,q,r}{\text{minimize}} && \|H(z) - \frac{p(z)}{q(z)} - \frac{r(1/z)}{q(1/z)}\|_\infty \\ & \text{subject to} && \deg(q) = m, \quad \deg(p) \leq m, \quad \deg(r) < m \\ & && q(z) \text{ is a Schur polynomial.} \end{aligned} \quad (2)$$

In program (2), an anti-stable rational part $\frac{r(1/z)}{q(1/z)}$ is added to the setup of (1) and because of these extra decision variables, program (2) is a relaxation of (1). Solving program (2), a (suboptimal) reduced model can simply be obtained as $\hat{H}(z) = \frac{p(z)}{q(z)}$. It can be shown that program (2) is quasi-convex after the re-parameterization to be discussed in the next subsection.

3.2 Re-parameterization of relaxation scheme

It is not convenient to directly work with program (2), as the set $\Omega_{qpr}^m := \{(q, p, r) \in \mathbb{R}^m \times \mathbb{R}^{m+1} \times \mathbb{R}^m\}$:

$$\begin{aligned} q(z) &= z^m + q_{m-1}z^{m-1} + \dots + q_0 \\ p(z) &= p_m z^m + p_{m-1}z^{m-1} + \dots + p_0 \\ r(z) &= r_{m-1}z^{m-1} + r_{m-2}z^{m-2} + \dots + r_0 \\ q(z) &\neq 0, \quad \forall z \in \mathbb{C} : |z| \geq 1 \end{aligned} \quad (3)$$

is not convex if $m > 2$. However, the following lemma states that a more convenient (i.e. quasi-convex), yet equivalent program exists.

LEMMA 3.1. *Let $m \in \mathbb{N}$. Let Ω_{qpr}^m be defined in (3). Let Ω_{abc}^m be the set of all $(a, b, c) \in \mathbb{R}^m \times \mathbb{R}^{m+1} \times \mathbb{R}^m$:*

$$\begin{aligned} a(z) &= (z^m + z^{-m}) + a_{m-1}(z^{m-1} + z^{-m+1}) + \dots + a_0 \\ b(z) &= b_m(z^m + z^{-m}) + b_{m-1}(z^{m-1} + z^{-m+1}) + \dots + b_0 \\ c(z) &= \frac{1}{j}(c_m(z^m - z^{-m}) + \dots + c_1(z - z^{-1})) \end{aligned} \quad (4)$$

satisfying

$$a(z) > 0 \quad \forall z \in \mathbb{C} : |z| = 1$$

Then there exists a one-to-one map $\tau_m : \Omega_{qpr}^m \mapsto \Omega_{abc}^m$:

$$H(e^{j\omega}) = \frac{p(e^{j\omega})}{q(e^{j\omega})} + \frac{r(e^{-j\omega})}{q(e^{-j\omega})} = \frac{b(e^{j\omega}) + jc(e^{j\omega})}{a(e^{j\omega})}, \quad \forall \omega \in [0, \pi]. \quad (5)$$

Given $(q, p, r) \in \Omega_{qpr}^m$, $\tau_m(q, p, r) \in \Omega_{abc}^m$ is defined by

$$\begin{aligned} a(z) &= q(z)q(1/z) \\ b(z) &= \frac{1}{2}[p(z)q(1/z) + q(z)r(1/z) + p(1/z)q(z) + q(1/z)r(z)] \\ c(z) &= \frac{1}{2j}[p(z)q(1/z) + q(z)r(1/z) - p(1/z)q(z) - q(1/z)r(z)]. \end{aligned}$$

Given $(a, b, c) \in \Omega_{abc}^m$, $(q, p, r) = \tau^{-1}(a, b, c) \in \Omega_{qpr}^m$ is defined by

$$q(z) = \prod_{z_k^* a(z_k) = 0, |z_k| < 1} (z - z_k), \quad (6)$$

and p, r are found as the unique solution to

$$p(z)q(1/z) + q(z)r(1/z) = b(z) + jc(z) \quad (7)$$

The implication of the lemma is that the non-convex stability constraint $q(z) \neq 0, \forall z \in \mathbb{C} : |z| \geq 1$ in (3) can be replaced by the convex (to be shown) positivity constraint $a(z) > 0, \forall z \in \mathbb{C} : |z| = 1$ and this paves way to the discovery of efficient algorithms for solving the relaxation problem. With the re-parameterization given by the previous lemma, positivity of $a(z)$ and applying Euler's formula, program (2) can equivalently be formulated as

$$\begin{aligned} & \underset{\tilde{a}, \tilde{b}, \tilde{c}, \gamma}{\text{minimize}} && \gamma \\ & \text{subject to} && |H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega)| < \gamma\tilde{a}(\omega), \quad \forall \omega \in [0, \pi], \\ & && \tilde{a}(\omega) > 0, \quad \forall \omega \in [0, \pi], \\ & && \deg(\tilde{a}) = m, \deg(\tilde{b}) \leq m, \deg(\tilde{c}) \leq m, \end{aligned} \quad (8)$$

with $\tilde{a}(\omega) = 1 + \tilde{a}_1 \cos(\omega) + \dots + \tilde{a}_m \cos(m\omega)$, $\tilde{b}(\omega) = \tilde{b}_0 + \tilde{b}_1 \cos(\omega) + \dots + \tilde{b}_m \cos(m\omega)$ and $\tilde{c}(\omega) = \tilde{c}_1 \sin(\omega) + \dots + \tilde{c}_m \sin(m\omega)$. It can be verified that program (8) is quasi-convex for the following reasons: 1) $\tilde{a}(\omega), \forall \omega \in [0, \pi]$ defines an intersection of infinite many halfspaces, and 2) the γ level set of the objective function is

$$\text{Re} \left(\theta (H(e^{j\omega})\tilde{a}(e^{j\omega}) - \tilde{b}(e^{j\omega}) - j\tilde{c}(e^{j\omega})) \right) < \gamma\tilde{a}(e^{j\omega}), \quad (9)$$

$\forall \omega \in [0, \pi], |\theta| \leq 1$, which is another intersection of halfspaces, parameterized by ω and θ . Therefore, program (8) is a quasi-convex program, and it is typically solved by the localization/cutting plane strategy such as the ellipsoid algorithm [2].

3.3 Constructing the reduced model

The denominator $q(z)$ and the numerator $p(z)$ of the reduced model can be found by applying eq. (6) and eq. (7) in Lemma 3.1, but this tends not to be numerically robust. Therefore, the following construction procedure is proposed instead. Once $q(z)$ is found, $p(z)$ is calculated as the optimal solution to the following program

$$\begin{aligned} & \underset{p,\gamma}{\text{minimize}} && \gamma \\ & \text{subject to} && |H(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})}| < \gamma, \quad \forall \omega \in [0, \pi], \\ & && \deg(p) \leq m. \end{aligned} \quad (10)$$

Note that program (10) is convex and can be solved by the localization methods.

4. CONSTRUCTING ORACLES

In applying the method of ellipsoids, the most important information that a user needs to supply is the functions that defines the target set (the set of all feasible points that attain the minimum of the objective function). These user supplied functions are typically referred to as oracles. Difficult-to-construct oracles will be discussed in this section.

4.1 Stability: Positivity constraint

From Lemma 3.1 it is shown that positivity constraint $\tilde{a}(\omega) > 0$ in program (8) is equivalent to stability constraint of $q(z)$ being a Schur polynomial in program (2). Therefore, the positivity constraint must be imposed strictly $\forall \omega \in [0, \pi]$ and therefore the common engineering practice of enforcing such constraint on only a finite set of points in that interval will not suffice. In order to address this issue consider the positivity constraint

$$\tilde{a}(\omega) = 1 + a_1 \cos(\omega) + \dots + a_m \cos(m\omega) > 0, \forall \omega \in [0, \pi]. \quad (11)$$

If $\exists \omega_0 \in [0, \pi] : \tilde{a}(\omega_0) = 0$, then $\tilde{a}(\omega_0) > 0$ defines a positivity cut, otherwise constraint (11) must be met because 1) $\exists \omega_1 \in [0, \pi] : \tilde{a}(\omega_1) > 0$ as implied by the fact that $\int_0^\pi \tilde{a}(\omega) d\omega > 0$ and 2) the continuous function $\tilde{a}(\omega)$ cannot be both positive and negative in between 0 and π without hitting 0. It can be verified that finding such ω_0 can be achieved by a simple root finding procedure of an ordinary polynomial of degree $2m$.

4.2 Passivity: Positive real constraint

For some applications it is desirable that the reduced model has positive real part. In order to impose this constraint, it suffices to note that real part of the relaxation in program (8) is $\tilde{b}(\omega)/\tilde{a}(\omega)$. Therefore, the only modification to (8) is to add the constraint $\tilde{b}(\omega) > 0, \forall \omega \in [0, \pi]$ and the treatment of this oracle is similar to that of the positivity constraint discussed in Subsection 4.1.

However, it should be noted that program (10) should be modified accordingly to guarantee the positive realness of the final reduced model. That is,

$$p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega}) > 0, \forall \omega \in [0, \pi]. \quad (12)$$

It is important to realize that constraint (12) is linear with respect to the decision variable $p(z)$ and the left side defines a trigonometric polynomial. As a result, when applying localization method to solve this program, the same oracle as the positive real part constraint can be used.

4.3 Passivity: Bounded real constraint

For S-parameter models, the notion of dissipative system is given by the bounded real condition (i.e. $|H(z)| < 1, \forall z \in \mathbb{C}, |z| = 1$). To model this property, program (8) can be modified by adding the constraint $\tilde{a}(\omega) > |\tilde{b}(\omega) + j\tilde{c}(\omega)|, \forall \omega \in [0, \pi]$. To construct the oracle, first check the positivity of the trigonometric polynomial $\tilde{a}(\omega)^2 - \tilde{b}(\omega)^2 - \tilde{c}(\omega)^2 > 0, \forall \omega \in [0, \pi]$. If this condition is met, then bounded realness is satisfied at the current query point, otherwise $\omega_0 \in [0, \pi]$ is found and $\tilde{a}(\omega_0) > |\tilde{b}(\omega_0) + j\tilde{c}(\omega_0)|$ defines a cut. It is noted that program (10) should be modified similarly to preserve the passivity of the final reduced model.

4.4 Explicit approximation of quality factor

We show here that the proposed method is quite flexible and additional constraints can be added to the optimization in order to address specific needs of circuit designers. For instance, when the transfer function H is the impedance of an RF inductor, the quality factor $Q(\omega) := \frac{\text{Im}(H(e^{j\omega}))}{\text{Re}(H(e^{j\omega}))}, \omega \in [0, \pi]$ is of critical importance for the system performance. In this case, the basic framework in (8) can be modified to guarantee a very good quality factor accuracy.

$$\begin{aligned} & \text{minimize } \gamma \\ & \quad \tilde{a}, \tilde{b}, \tilde{c}, \gamma \\ & \text{subject to } |H(e^{j\omega})\tilde{a}(\omega) - \tilde{b}(\omega) - j\tilde{c}(\omega)| < \gamma\tilde{a}(\omega), \\ & \quad \left| \frac{\text{Im}(H(e^{j\omega}))}{\text{Re}(H(e^{j\omega}))} \tilde{b}(\omega) - \tilde{c}(\omega) \right| < \rho\gamma\tilde{b}(\omega), \\ & \quad \tilde{a}(\omega) > 0, \tilde{b}(\omega) > 0, \quad \forall \omega \in [0, \pi], \\ & \quad \deg(\tilde{a}) = m, \deg(\tilde{b}) \leq m, \deg(\tilde{c}) \leq m. \end{aligned} \quad (13)$$

ρ in the second set of constraint is a tuning parameter of relative accuracy between match on frequency response and quality factor.

The oracles for program (13) are similar to those for program (8) with the positive real part constraint and the reduced model should be constructed using

$$\begin{aligned} & \text{minimize } \gamma \\ & \quad p, \gamma \\ & \text{subject to } |H(e^{j\omega}) - \frac{p(e^{j\omega})}{q(e^{j\omega})}| < \gamma, \forall \omega \in [0, \pi], \\ & \quad \left| \frac{\text{Im}(H(e^{j\omega}))}{\text{Re}(H(e^{j\omega}))} - \frac{p(e^{j\omega})q(e^{-j\omega}) - p(e^{-j\omega})q(e^{j\omega})}{p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega})} \right| < \rho\gamma \\ & \quad p(e^{j\omega})q(e^{-j\omega}) + p(e^{-j\omega})q(e^{j\omega}) > 0, \forall \omega \in [0, \pi]. \\ & \quad \deg(p) \leq m, \end{aligned} \quad (14)$$

Again, this program is quasi-convex and the oracle procedure with constraint (12) can be applied here as well.

5. EXTENSION TO PMOR

This section discusses how the framework in (8) can be extended to solve the problem of PMOR (i.e. to construct a parameter dependent model guaranteed stable for all parameters of interest). In particular, the positivity (stability) oracle will be discussed.

The idea of enabling PMOR capability of program (1) is to allow the coefficients of the trigonometric polynomials to be parameter dependent. That is,

$$\begin{aligned} a(z, p) &= a_0(p) + a_1(p)(z + z^{-1}) + \dots + a_m(p)(z^m + z^{-m}), \\ b(z, p) &= b_0(p) + b_1(p)(z + z^{-1}) + \dots + b_m(p)(z^m + z^{-m}), \\ c(z, p) &= \frac{1}{j}(c_1(p)(z - z^{-1}) + \dots + c_m(p)(z^m - z^{-m})), \end{aligned}$$

where $p \in \mathcal{P} \subset \mathbb{R}^{n_p}$ denoting that the set of all vector of design parameters p is \mathcal{P} . The parameterized version of program (8) is

$$\begin{aligned} & \text{minimize } \gamma \\ & \quad \tilde{a}, \tilde{b}, \tilde{c}, \gamma \\ & \text{subject to } |H(e^{j\omega}, p)\tilde{a}(\omega, p) - \tilde{b}(\omega, p) - j\tilde{c}(\omega, p)| < \gamma\tilde{a}(\omega, p), \\ & \quad \forall \omega \in [0, \pi], \forall p \in \mathcal{P}, \\ & \quad \tilde{a}(\omega, p) > 0, \quad \forall \omega \in [0, \pi], p \in \mathcal{P} \\ & \quad \deg(\tilde{a}) = m, \deg(\tilde{b}) \leq m, \deg(\tilde{c}) \leq m. \end{aligned} \quad (15)$$

The only problem with applying the localization methods to program (15) is that the positivity oracle is different if \mathcal{P} is not a finite set. In general, it would be very difficult to ensure the parameterized positivity constraint if the parameter dependence of \tilde{a} on p is arbitrary. However, if the dependence is polynomial

$$\tilde{a}(\omega, p) = \sum_k \left(\sum_i c_{i_1, i_2, \dots, i_{n_p}} p_1^{i_1} p_2^{i_2} \dots p_{n_p}^{i_{n_p}} \right) \cos(k\omega), \quad (16)$$

then it can be shown that a sufficient condition for positivity can be found using the sum-of-squares (SOS) argument and the resulting oracle will involve solving a semidefinite program, which is solvable in polynomial time. The underlying reason is as follows. If \mathcal{P} is bounded, then $p_k = \frac{\bar{p}_k + p_k}{2} + (\frac{\bar{p}_k - p_k}{2}) \cos(\omega_k)$ and therefore $\tilde{a}(\omega, p)$ is a trigonometric polynomial (in more than one variable). Therefore, $\tilde{a}(\omega, p) > 0$ iff it is a sum of squares (SOS) of trigonometric polynomials, as stated in the following theorem [16].

THEOREM 5.1. *A trigonometric polynomial is positive if and only if it is a finite sum of squares of trigonometric polynomials.*

It can be shown that an SOS of trigonometric polynomials is equivalent to a quadratic form $\theta'Q\theta$, where θ is a vector of appropriately chosen monomials and $Q = Q' \geq 0$. Let $p = p(\omega_p)$ as $\omega_p(k) = \omega_k$ in $p_k = \frac{\bar{p}_k + p_k}{2} + (\frac{\bar{p}_k - p_k}{2}) \cos(\omega_k), \forall k$, then the procedure described in Algorithm 1 can be used as a stability oracle.

Denoting x_a as the coefficient vector of \tilde{a} , the following lemma certifies the correctness of the oracle.

LEMMA 5.1. *If the optimal value of program (17) $y^* < 0$, then $\tilde{a}(\omega, p) > 0, \forall \omega \in [0, \pi], p \in \mathcal{P}$. Otherwise, a cut $(\alpha, \beta) \in \mathbb{R}^{n_a} \times \mathbb{R}$*

Algorithm 1: PMOR POSITIVITY ORACLE**Input:** query point \tilde{a} **Output:** declaration of constraint met or a cut (α, β) PMOR POS(\tilde{a})

- (1) Given trigonometric polynomial $\tilde{a}(\omega, p)$, pick an “appropriate” vector of monomials $\theta \in \mathbb{C}^k$
- (2) Solve the SDP

$$\begin{aligned} & \underset{y \in \mathbb{R}, P}{\text{minimize}} && y \\ & \text{subject to} && \theta' P \theta = \tilde{a}(z, \omega_p) + y, \forall |z| = 1, \forall \omega_p \in [0, \pi] \\ & && P = P' \geq 0. \end{aligned} \quad (17)$$

- (3) **if** optimal $y^* < 0$
- (4) **return** Positivity constraint is met
- (5) **else**
- (6) **return** Cut (α, β)

can be returned such that $\alpha' x_a > \beta$, for all x_a such that optimal objective value of program (17) is negative, thus constituting a (restrictive) positivity cut.

It must be noted that the cut returned by program (17) is restrictive in the sense that it eliminates all the options that do not result in $y^* < 0$, but some of which can still be positive trigonometric polynomials. Nevertheless, it is generally true that this is not too conservative if the vector of monomials is chosen properly.

While the specific construction of the positivity constraint oracle in Lemma 5.1 requires the dependence of \tilde{a} on the design parameter to be polynomial, there is no restriction in the dependence of \tilde{b} and \tilde{c} and they can be chosen to best fit the problem at hand. Finally it is noted that program (17) can be solved using free SDP solver like SeDuMi [21].

6. ALGORITHM SUMMARY

This section gives a summary of how the proposed optimization framework could be used for MOR and PMOR. For both of these algorithms, it is assumed, without loss of generality that the original system is specified as a transfer function or as measurement data evaluated by a field solver or available.

Algorithm 2: MOR**Input:** $H(z)$ **Output:** $\hat{H}(z)$

- (1) Solve program (8) using ellipsoid algorithm to obtain the relaxation $(\tilde{a}, \tilde{b}, \tilde{c})$
- (2) Compute denominator $q(z)$ using eq. (6)
- (3) Solve program (10) to obtain numerator $p(z)$
- (4) Synthesize a state space realization of the reduced model $\hat{H}(z) = p(z)/q(z)$. See [11] for detail.

The algorithm (MOR) given serves as the basic framework, but it can be modified to account for several additional desirable constraints, as discussed in Section 4. For instance, the Algorithm 3 implements a PMOR procedure, and it is specialized in the case where the full model has only one pair of “dominant poles”. It is given because it can take advantage of the problem specific insight common in RF inductor design. Note that the reduced model $\hat{H}(z, p)$ is stable because $|\hat{z}^*(p)| < 1$ as constructed and $\hat{H}(z, p)$ is guaranteed stable $\forall p \in \mathcal{P}$.

6.1 Complexity

The complexity of the method can roughly be divided into two parts. The first part is the computation of the frequency samples,

Algorithm 3: PMOR: RF INDUCTOR DESIGN**Input:** $H(z, p)$ **Output:** $\hat{H}(z, p)$

- (1) Construct reduced models $\tilde{H}_p(z)$ for each $p \in \mathcal{P}_1 \subset \mathcal{P}$, where \mathcal{P}_1 is a finite set
- (2) Identify the dominant poles z_p^* of models $\tilde{H}_p(z)$
- (3) Construct proper “non-dominant” systems $H_p^1(z)$:

$$\tilde{H}_p(z) = \frac{K_p}{(z - z_p^*)(z - \bar{z}_p^*)} (z^2 + A_p z + H_p^1(z)), \quad (18)$$

where $K_p \in \mathbb{R}$ and $A_p \in \mathbb{R}$

- (4) Construct global interpolation model $\hat{K}(p)$, $\hat{A}(p)$ and $\hat{z}^*(p)$. Special attention should be paid to the model $\hat{z}^*(p)$ to make sure that $|\hat{z}^*(p)| < 1, \forall p \in \mathcal{P}$
- (5) Solve program (15) to find a parameterized model $\hat{H}^1(z, p)$ with non-dominant systems $H_p^1(z)$ as inputs.
- (6) Construct reduced model of the original system using eq. (18). That is,

$$\hat{H}(z, p) = \frac{\hat{K}(p)}{(z - \hat{z}^*(p))(z - \bar{\hat{z}}^*(p))} (z^2 + \hat{A}(p)z + \hat{H}^1(z, p)).$$

which, when using accelerated solvers, is $O(n \log(n))$ for each frequency point with n being the order of the full model. The second part is the cost of the optimization algorithm, which is $O(q^6)$ with q being the order of the reduced model, when using the method of ellipsoids. Therefore, construction of relatively high order reduced model (e.g. order > 100) is not feasible.

7. APPLICATIONS AND EXAMPLES

In this section several application examples are shown to illustrate how the proposed optimization based model reduction algorithm works and performs in practice.

7.1 Comparison with PRIMA

RF inductor example. The first example is a comparison between multi-point moment matching (PRIMA) [17] and the proposed algorithm for reducing a spiral RF inductor model generated by an electromagneto quasi-static (EMQS) mixed potential integral equation (MPIE) solver [12]. The original model has order 1576. PRIMA is set to match 2 moments at DC, 6 moments at 4,8,12 GHz. The resulting model has order 20. On the other hand, two models are constructed using the proposed method. One has order 14 using 20 frequency samples (same computational cost as PRIMA), and the other has order 20 using 40 frequency samples (same order as PRIMA). Two kinds of error metrics are computed: $\max(\frac{|H(f) - \hat{H}(f)|}{|H(f)|})$, $f \in [0, 14\text{GHz}]$ and $\frac{\max(Q(f)) - \max(\hat{Q}(f))}{\max(Q(f))}$, where $Q(f)$ is the quality factor defined in Subsection 4.4. Comparison results are shown in Table 1.

Table 1: Reduction of RF inductor using QCO and MM

	QCO	QCO	MM
order	14	20	20
cost (# of solves)	20	40	20
error (%) : H	6.9×10^{-3}	7.1×10^{-4}	1.8×10^{-3}
error (%) : Q peak	1.1×10^{-3}	3.3×10^{-4}	8.8×10^{-5}

RLC line example. The next example is to reduce an RLC line of 10 sections (full model order 20) with an open circuit termination. The transfer function is the admittance. The model is ob-

tained as follows: inductor currents and capacitor voltages are the state variables. The KCL is imposed at each capacitor node and the branch equation is used between adjacent nodes. The reduced models of both methods have order 10 and PRIMA is set to match 4 moments at 10^4 rad/s, 4 moments at 5×10^4 rad/s and 2 moments at 10^5 rad/s. Figures 1 and 2 show the magnitudes of the admittance of the full model and the reduced models by PRIMA and the proposed method, respectively. The difficulties encountered when modelling this example with PRIMA are discussed in [22].

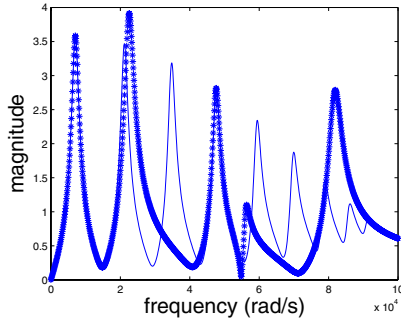


Figure 1: Magnitude of admittance of an RLC line. Solid: full model. Solid with stars: PRIMA 10th order ROM.

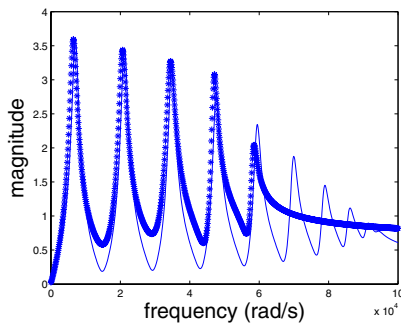


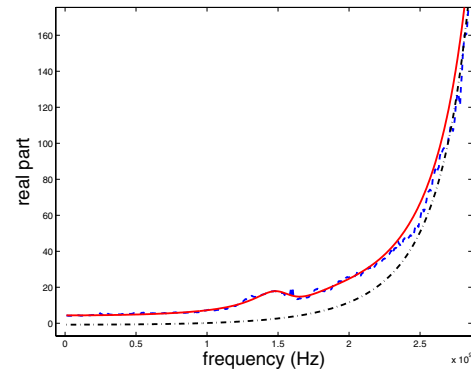
Figure 2: Magnitude of admittance of an RLC line. Solid: full model. Solid with stars: QCO 10th order ROM.

7.2 Comparison with a rational fit algorithm

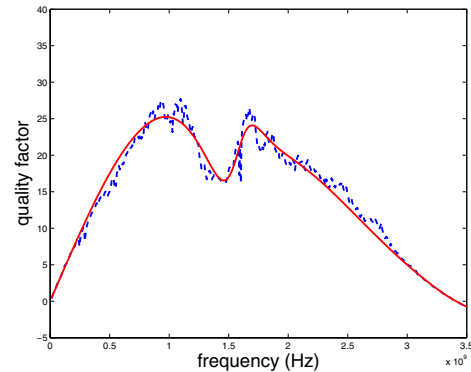
In the third example we compare the proposed method with an existing optimization based rational fit [4] by constructing a reduced model from measured frequency response of a fabricated spiral RF inductor [18]. In this example, the order of the reduced model is 10 and the positive real part constraint is imposed, the quality factor is explicitly minimized. That is, program (13) is solved with tuning parameter $\rho = 10^{-4}$. Figure 3.b shows the quality factor of the reduced model compared to the measured data. Figure 3.a compares the proposed approach to a model of the same order (10) generated using the optimization based approach in [4].

7.3 Comparison to measured S-parameters from an industry provided example

In the fourth example we identify a reduced model from measured multi-port S-parameter data. Figure 4 shows the comparison result for one of the ports. The reduced model is order 20. The model was identified in 30 seconds on a matlab implemented version of our algorithm running on a Pentium 4 with 1GHz clock.



(a) real part of impedance



(b) quality factor

Figure 3: Identification of RF inductor. Dash line: measurement. Solid line: proposed reduced model. Lower dash-dot line only for (a): approach in [4]

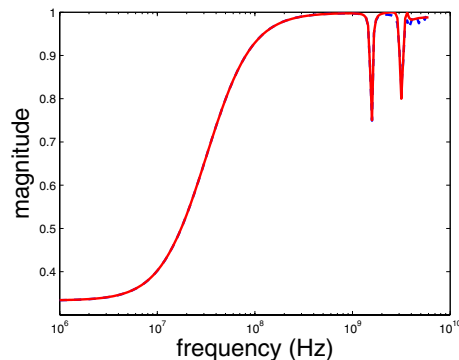


Figure 4: Magnitude of one of the port S-parameters for an industry provided example. Solid line: reduced model (order 20). Dash line: measured data (almost overlapping).

7.4 Frequency dependent matrices example

In the fifth example we apply the proposed method to reduce a model of an RF inductor generated by a full wave MPIE solver accounting for the substrate effect using layered Green's functions [10]. Since the system matrices are frequency dependent, the order of the full model is infinite. The order of the reduced model is 6 and the positive real part constraint is imposed. Figure 5 shows the result of the quality factor.

7.5 PMOR example

In the sixth example we construct a parameterized reduced model

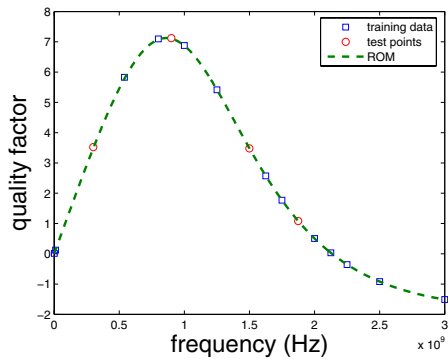


Figure 5: Quality factor of an RF inductor with substrate captured by layered Green's function. Full model is infinite order and ROM order is 6.

for a 7 turn spiral RF inductor (full model generated by an EMQSMPIE solver [12]) whose wire width W and wire separation D are allowed to vary in the range of $(W, D) \in [1\mu\text{m}, 20\mu\text{m}]$. Algorithm 3 described in Section 6 is applied. In order to construct the reduced model, 270 systems in the design parameter space are chosen as "training points". The example then tests the result for different sets of (W, D) : one with $W \equiv 16.5\mu\text{m}$ and the other with $W \equiv 19\mu\text{m}$.

Finally, Figure 7 shows the matching of the frequency of the peak of the quality factor. It is interesting to note that the peak is *not* monotonically increasing as D increases. This phenomenon, which is accurately captured by the proposed algorithm, disappears when the bridgewire is moved further away from the turn.

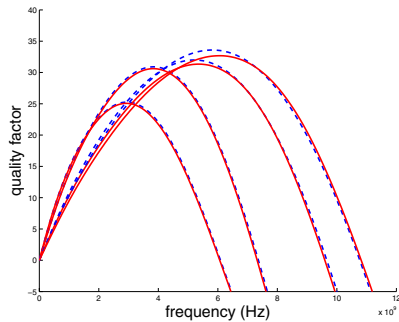


Figure 6: Quality factor for $W=16.5\mu\text{m}$, $D=1\mu\text{m}, 5\mu\text{m}, 18\mu\text{m}, 20\mu\text{m}$. Solid line: parameterized reduced model. Dash line: full model.

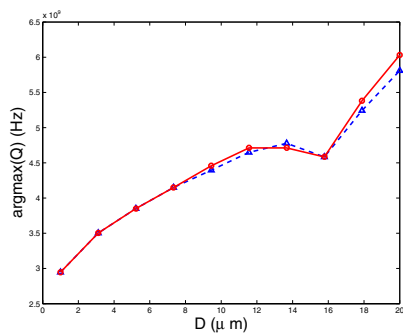


Figure 7: Frequency of the peak of the quality factor for $W=16.5\mu\text{m}$. Solid line: parameterized reduced model. Dash line: full model.

8. CONCLUSION

In this paper a relaxation framework to the optimal \mathcal{H}_∞ MOR problem is proposed. The framework has been demonstrated to perform approximately as well as PRIMA when reducing large systems and better than PRIMA for an RLC line example. Unlike PRIMA, the proposed method can reduce models with frequency dependent system matrices. Unlike other optimization based methods, the proposed method has been shown to be very flexible in preserving stability and passivity. Finally, the proposed optimization setup has also been extended to parameterized MOR problems. Several examples have been presented validating the MOR approach on measured data and the PMOR approach on a large RF inductor.

This work is supported by the Gigascale Systems Research Center and the Semiconductor Research Corporation. The authors would like to thank Dmitry Vasilyev for providing the RLC line example in Subsection 7.1.

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