A Convex Programming Approach to Positive Real Rational Approximation

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ABSTRACT

As system integration evolves and tighter design constraints must be met, it becomes necessary to account for the non-ideal behavior of all the elements in a system. Certain devices common in high-frequency integrated circuit applications, such as spiral inductors, SAW filters, etc, are often described and studied in the frequency domain. Models take the form of frequency domain data obtained through measurement or through physical simulation. Usually the available data is sampled, incomplete, noisy, and covers only a finite range of the spectrum.

In this paper we present a methodology for generating guaranteed passive time-domain models of frequency-described subsystems. The methodology presented is based on convex programming based algorithms for fixed denominator system identification. The algorithm is *guaranteed* to produce a passive system model that is optimal in the sense of having minimum weighted square error in the frequency band of interest over all models with a prescribed set of system poles. An incremental-fitting reformulation of the problem is also introduced that trades optimality for efficiency while still guaranteeing passivity. Results of the application of the proposed methodologies to the modeling of a variety of subsystems are presented and discussed.

1 INTRODUCTION

Especially in high-frequency applications, certain devices are usually described and studied in the frequency domain. Devices such as coil inductors, SAW filters, non-ideal transmission lines and high-frequency transistors are commonly described by manufacturers and designers by their frequency dependent scattering parameter or admittance matrices. Frequency domain data is either obtained through measurement or through physical simulation. In either case, the available data is sampled, incomplete, noisy, and covers only a finite range of the spectrum.

It is not trivial to generate accurate circuit-level models for all the devices in a system. Such models are however necessary for the time domain simulation of larger designs and to account for the non-ideal characteristics of the devices. Although harmonic balance simulators can handle devices described by their frequency response, they cannot adequately treat highly non-linear designs such as oscillators and mixers. On the other hand, time-domain simulators, using state-space model integration, that can deal with high-order non-linearities, require time-domain models. It is necessary that these models have frequency responses that match the available data. The models must also possess stability and passivity properties similar to those of the physical system that they represent. One approach to time-domain simulation of frequency-described devices is to synthesize a state space model whose frequency behavior approximates the original device in the frequency band of interest. Many approaches have been proposed for the related problems of model generation, rational approximation, and system identification [19, 14, 13, 5, 12, 15, 16, 17, 18, 10, 8]. The difficulty is that no single scheme that we are aware of can simultaneously satisfy all the constraints on a practical method, particularly in a design automation environment where engineers who wish to generate models from measured data are not likely to be experts in rational approximation.

Previously, we have addressed the robust generation of accurate stable models [10]. However, such models were not guaranteed to be passive. Passivity is an important property of certain physical systems. Networks composed by resistors, capacitors and inductors are passive, they do not generate energy. Systems that always consume energy are called strictly passive. Brune [7] proved that the admittance and impedance parameter matrices of passive electrical networks are positive real matrix rational functions. The matrix transfer function of a linear time-invariant system is positive real if its state space realization satisfies the Positive Real Lemma [2]. Interconnected passive systems are passive. Stable systems do not possess this closure property. Incorrectly representing a passive device by a merely stable model may lead to the instability of the overall system and to incorrect simulation results.

In this paper, we assume that a stable matrix rational function model already exists. Such a model can readily be generated using previously developed rational approximation algorithms [10, 16, 18, 8]. The matrix numerator is then recalculated such that the Positive Real Lemma is satisfied and the error between the data set and the model's frequency response is minimized. This is accomplished by solving a linear program subject to a set of linear matrix inequality constraints. Since the objective function and the constraints are convex functions of the optimization variables, efficient convex programming algorithms can be applied [6]. Apart from failure due to convergence or numerical problems, these algorithms are guaranteed to find an optimal solution.

This paper is structured as follows. In the next section, background information on passive systems and positive realness is presented. In section 3, model representation is considered. In section 4, the positive real constrained multivariable fitting problem is formulated as a linear problem subject to a set of linear matrix inequality constraints. Experimental results regarding computational complexity are also presented. In section 5, an incremental fitting strategy that reduces the dependency of the computational cost with the number of ports is proposed. Certain numerical issues are addressed in section 6. In section 7, experimental results illustrating the use of the proposed formulations to generate models for several real-world examples are presented. Final conclusions are drawn in section 8.

2 BACKGROUND

Passivity is an important property of certain physical systems. Networks composed by resistors, capacitors and inductors are passive, they do not generate energy. Systems that always consume energy are called strictly passive. Interconnected (strictly) passive systems are (strictly) passive. Stable systems do not possess this closure property. Stable systems loaded by stable, and even passive systems, may not even constitute an overall stable system. Therefore, it is important that passive models represent passive systems.

Passive systems can be characterized in the time-domain and in the frequency domain. Brune [7] proved that the admittance and impedance parameter matrices of passive electrical networks are positive real matrix rational functions.

A square matrix function $\mathbf{H}(s)$ is said to be positive real if it satisfies

$$\mathbf{H}(s) \text{ is analytic, for } \mathrm{Re}[s] > 0. \quad (1)$$
$$\overline{\mathbf{H}(s)} = \mathbf{H}(\bar{s}), \text{ for } \mathrm{Re}[s] > 0. \quad (2)$$
$$\mathbf{H}(s) + \mathbf{H}(s)^H \geq \mathbf{0}, \text{ for } \mathrm{Re}[s] > 0. \quad (3)$$

The matrix transfer function of a linear time-invariant system is positive real if its state space realization satisfies the Positive Real Lemma.

Theorem 2.1 (Positive Real Lemma [2]) *Let $\mathbf{H}(s)$ be a matrix transfer function such that all its poles lie either on the left-half plane or on the imaginary axis, in which case they are simple. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a minimal realization of $\mathbf{H}(s)$. Then $\mathbf{H}(s)$ is positive real if and only if there exists a symmetric positive definite matrix $\mathbf{K}$ and matrices $\mathbf{W}$ and $\mathbf{L}$ such that the following Lur'e equations hold*

$$\begin{aligned} \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} &= -\mathbf{L}\mathbf{L}^T & (4) \\ \mathbf{K}\mathbf{B} &= \mathbf{C}^T - \mathbf{L}\mathbf{W} & (5) \\ \mathbf{D} + \mathbf{D}^T &= \mathbf{W}^T\mathbf{W} & (6) \end{aligned}$$

For a proof see [2].

If $\mathbf{D} + \mathbf{D}^T > 0$, this condition is equivalent to requiring that there exists $\mathbf{K} = \mathbf{K}^T \geq 0$ such that

$$\begin{bmatrix} -\mathbf{A}^T\mathbf{K} - \mathbf{K}\mathbf{A} & -\mathbf{K}\mathbf{B} + \mathbf{C}^T \\ -\mathbf{B}^T\mathbf{K} + \mathbf{C} & \mathbf{D} + \mathbf{D}^T \end{bmatrix} \geq 0 \quad (7)$$

is satisfied.

If $\mathbf{D} = \mathbf{0}$, it follows from Theorem 2.1 that $\mathbf{H}(s)$ is positive real if and only if there exists $\mathbf{K}^T = \mathbf{K} \geq 0$ such that

$$\begin{aligned} \mathbf{A}^T\mathbf{K} + \mathbf{K}\mathbf{A} &\leq 0 & (8) \\ \mathbf{B}^T\mathbf{K} &= \mathbf{C}. & (9) \end{aligned}$$

For further references on passivity and the more general concept of dissipativity see, for example [2, 20, 3, 4, 11].

3 MODEL REPRESENTATION

The methodology we will now describe for generating guaranteed passive time-domain models of frequency-described subsystems is based on convex programming based algorithms for fixed denominator system identification. It is therefore assumed that a stable matrix rational function (i.e. a state-space model with no poles in the right half of the complex plane) has already been generated, and that the pole estimates are sufficiently good for the level of approximation accuracy desired. As mentioned before, any convenient rational approximation algorithm can be used to generate this initial approximation [10, 16, 18, 8]. In previous work [10], we demonstrated numerically robust algorithms that could generate high-order models with a reasonable amount of computational effort. Even though such methods were directly applicable to either SISO (single-input, single-output) or MIMO (multiple-input, multiple-output) systems, we found that in general for MIMO systems a good compromise between robustness, accuracy and model size, was to approach them as a set of SIMO systems. This is also the approach we follow in this work.

In the following, a multivariable n-port system is therefore modeled by the concatenation of n single-input multiple-output state space models as illustrated in Eqn. (10).

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_{n,n} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{B}_{n,n} \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{1,1} & \cdots & \mathbf{C}_{1,n} \\ \vdots & & \vdots \\ \mathbf{C}_{n,1} & \cdots & \mathbf{C}_{n,n} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_{1,1} & \cdots & \mathbf{D}_{1,n} \\ \vdots & & \vdots \\ \mathbf{D}_{n,1} & \cdots & \mathbf{D}_{n,n} \end{bmatrix} \quad (10)$$

It is assumed that an estimate of the poles of each subsystem is available. Using this information, a real block diagonal system matrix $\mathbf{A}_{k,k}$ is constructed for each n by 1 subsystem model. It is assumed that the poles are associated in n_b sets. Complex conjugate pairs and repeated poles belong to the same set. To each set, corresponds a block diagonal entry in $\mathbf{A}_{k,k}$. A state space representation for the k^{th} subsystem is given by

$$\mathbf{A}_{k,k} = \begin{bmatrix} \mathbf{A}_{k,k}^{(1)} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{A}_{k,k}^{(n_b)} \end{bmatrix} \quad \mathbf{B}_{k,k} = \begin{bmatrix} \mathbf{B}_{k,k}^{(1)} \\ \vdots \\ \mathbf{B}_{k,k}^{n_b} \end{bmatrix}$$
$$\mathbf{C}_{:,k} = \begin{bmatrix} \mathbf{C}_{k,k}^{(1)} & \cdots & \mathbf{C}_{k,k}^{(n_b)} \end{bmatrix} \quad \mathbf{D}_{:,k} = \sum_{q=1}^{n_b} \mathbf{D}_{k,k}^{(q)}. \quad (11)$$

For simplicity, in the following, the subsystem related notation is dropped.

The frequency response of the subsystem's model may be represented as a sum of strictly proper n by 1 vector rational functions and a constant real n by 1 vector term

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{D} + \textstyle\sum_{k=1}^{n_b} \mathbf{C}^{(k)} \left(\mathbf{I}s - \mathbf{A}^{(k)}\right)^{-1} \mathbf{B}^{(k)} \\ &= \mathbf{D} + \textstyle\sum_{k=1}^{n_b} \mathbf{R}^{(k)}(s) \\ &= \mathbf{D} + \textstyle\sum_{k=1}^{n_b} \mathbf{p}^{(k)}(s)/q^{(k)}(s)\end{aligned} \tag{12}$$

Let $s_1^{(k)}, \ldots, s_{n^{(k)}}^{(k)}$ represent the k^{th} set of poles and $n^{(k)}$ the number of elements it contains. The denominator polynomial

$$q^{(k)}(s) = \prod_{q=1}^{n^{(k)}} (s - s_q^{(k)}) = \sum_{q=0}^{n^{(k)}} a_q^{(k)} s^q \tag{13}$$

has real coefficients $a_0^{(k)}, \ldots, a_{n^{(k)}}^{(k)}$ where $a_{n^{(k)}}^{(k)} = 1$. Each rational function $\mathbf{R}^{(k)}(s)$ may be represented as

$$\mathbf{R}^{(k)} = \frac{\mathbf{p}^{(k)}(s)}{q^{(k)}(s)} = \frac{\sum_{q=0}^{n^{(k)}-1} \mathbf{C}_q^{(k)} s^q}{\sum_{q=0}^{n^{(k)}} a_q^{(k)} s^q}, \tag{14}$$

where $\mathbf{C}_q^{(k)}$ is an n by 1 real vector.

As repeated poles are uncommon, the number of poles in each set is usually small. The most frequent block size is two, corresponding to complex conjugate pole pairs. For such small blocks, no numerical problems arise from the use of the controller canonical representation. The choice of the controller canonical representation is also motivated by the fact that the positive real lemma inequality (7) is affine in the pair $(\mathbf{C}, \mathbf{K})$ but nonlinear in the pair $(\mathbf{B}, \mathbf{K})$. By using the controller canonical representation, the numerator coefficients, which are the problem's optimization variables, are the elements of $\mathbf{C}$. We note however that since the algorithm makes no assumptions about the specific structure of $\mathbf{A}$, any other "controller-like" representation would be a good choice, as long as the elements of $\mathbf{C}$ are chosen as the optimization variables. In particular one could use one of the orthogonal polynomial forms from [10] which could perhaps lead to a simplified structure for K.

4 PROBLEM FORMULATION

Assume that we are given admittance or impedance parameter matrices of a system $\widetilde{\mathbf{H}}(s)$ for a set of N frequency points, s_1 through s_N and that a stable matrix rational function model already exists. The problem at hand is to determine $\mathbf{C}$ such that the error function

$$E = \sum_{q=1}^{n} \sum_{p=1}^{n} \sum_{k=1}^{N} w_{k,p,q} \left\| \mathbf{H}_{p,q}(s_k) - \widetilde{\mathbf{H}}_{p,q}(s_k) \right\|_2^2, \tag{15}$$

where $\mathbf{H}_{p,q}(s) = \mathbf{C}_{p,q}(\mathbf{I}s - \mathbf{A}_{q,q})^{-1}\mathbf{B}_{q,q} + \mathbf{D}_{p,q}$, is minimized. Furthermore, it is required that the model have a positive real frequency response.

Generalizations such as the projection of the error on different spaces may also be considered. In the following, for simplicity, it is assumed that $w_{k,p,q} = 1$ for all values of k, p and q. In the results Section 7, other values, such as $w_{k,p,q} = 1/|\widetilde{\mathbf{H}}_{p,q}(s_k)|$, are used.

For a scalar rational function, with $\mathbf{D} > 0$, the constrained fitting problem may be stated as,

$$\begin{aligned} &\text{minimize} \quad \textstyle\sum_{p=1}^{n} \sum_{q=1}^{n} t_{p,q} \\ &\text{subject to} \begin{bmatrix} -\mathbf{A}^T\mathbf{K} - \mathbf{K}\mathbf{A} & -\mathbf{K}\mathbf{B} + \mathbf{C}^T \\ -\mathbf{B}^T\mathbf{K} + \mathbf{C} & \mathbf{D} + \mathbf{D}^T \end{bmatrix} \geq 0 \end{aligned} \tag{16}$$

$$\mathbf{K} = \mathbf{K}^T \geq 0, \tag{17}$$

$$(\mathbf{F}_{p,q}\mathbf{C}_{p,q}^T - \mathbf{G}_{p,q})^T(\mathbf{F}_{p,q}\mathbf{C}_{p,q}^T - \mathbf{G}_{p,q}) \leq t_{p,q} \tag{18}$$

$$\text{for } 1 \leq p, q \leq n.$$

where $\mathbf{F}_{p,q} \in \mathbb{R}^{2N \times n(p)}$ and $\mathbf{G}_{p,q} \in \mathbb{R}^{2N}$ are defined as (for the kth row),

$$\mathbf{F}_{p,q}(k,:) = \begin{cases} w_{p,q,k}\mathrm{Re}[\mathbf{B}_{q,q}^T(\mathbf{I}s_k - \mathbf{A}_{q,q}^T)^{-1}] & k \leq N \\ w_{p,q,k-N}\mathrm{Im}[\mathbf{B}_{q,q}^T(\mathbf{I}s_{k-N} - \mathbf{A}_{q,q}^T)^{-1}] & k > N \end{cases}$$

$$\mathbf{G}_{p,q}(k) = \begin{cases} w_{p,q,k}\mathrm{Re}[\widetilde{\mathbf{H}}_{p,q}(s_k) - \mathbf{D}_{p,q}] & k \leq N \\ w_{p,q,k-N}\mathrm{Im}[\widetilde{\mathbf{H}}_{p,q}(s_{k-N}) - \mathbf{D}_{p,q}] & k > N \end{cases}$$

If $\mathbf{D} = \mathbf{0}$, the inequality (16) should be replaced by

$$-\mathbf{A}^T\mathbf{K} - \mathbf{K}\mathbf{A} \geq 0 \text{ and } \mathbf{K}\mathbf{B} = \mathbf{C}^T. \tag{19}$$

If $\mathbf{D}$ is also considered to be an optimization variable, the additional constraint $\mathbf{D} + \mathbf{D}^T \geq 0$ should be introduced. To avoid having to present equations for $\mathbf{D}^T + \mathbf{D} \geq 0$ and $\mathbf{D} = \mathbf{0}$, it is assumed that $\mathbf{D}$ has already been estimated and that it is a positive definite matrix.

Since both the objective function and the constraints are convex functions of t, $\mathbf{K}$ and $\mathbf{C}$, it is possible to find a global minimum by using convex optimization algorithms. The next steps transform each quadratic inequality (18), in a well conditioned linear matrix inequality (LMI). A LMI is condition on the eigenvalues of a Hermitian matrix that is an affine function of the optimization variables. It has recently been proved that these problems can be efficiently solved in polynomial time (see [6] for references).

In a first step, each $\mathbf{F}_{p,q}$ is QR factored such that $\mathbf{F}_{p,q} = \mathbf{Q}_{p,q}\mathbf{R}_{p,q}$ where $\mathbf{Q}_{p,q}^T\mathbf{Q}_{p,q} = \mathbf{I}$ and $\mathbf{R}_{p,q}$ is upper triangular. Then $\widetilde{\mathbf{C}} = \mathbf{C}\mathbf{R}^T$ is defined. The product (18), is then carried out and $\mathbf{C}$ is replaced by $\widetilde{\mathbf{C}}\mathbf{R}^{-T}$, resulting in

$$\widetilde{\mathbf{C}}_{p,q}\widetilde{\mathbf{C}}_{p,q}^T - 2\mathbf{G}_{p,q}^T\mathbf{Q}_{p,q}\widetilde{\mathbf{C}}_{p,q}^T + \mathbf{G}_{p,q}^T\mathbf{G}_{p,q} < t_{p,q}. \tag{20}$$

By using the Schur complement[1] the equivalent linear matrix inequality

$$\begin{bmatrix} t + 2\mathbf{G}_{p,q}^T\mathbf{Q}_{p,q}\widetilde{\mathbf{C}}_{p,q}^T - \mathbf{G}_{p,q}^T\mathbf{G}_{p,q} & \widetilde{\mathbf{C}}_{p,q} \\ \widetilde{\mathbf{C}}_{p,q}^T & \mathbf{I} \end{bmatrix} \geq 0, \tag{22}$$

is obtained.

The formulation of the multi-port constrained fitting problem as a linear optimization problem subject to a set of linear matrix inequalities, and positive definite $\mathbf{D}$, becomes

$$\begin{aligned} &\text{minimize } \textstyle\sum_{p=1}^{n} \sum_{q=1}^{n} t_{p,q} \\ &\text{subject to} \begin{bmatrix} -\mathbf{A}^T\mathbf{K} - \mathbf{K}\mathbf{A} & -\mathbf{K}\mathbf{B} + \mathbf{R}^{-1}\widetilde{\mathbf{C}}^T \\ -\mathbf{B}^T\mathbf{K} + \widetilde{\mathbf{C}}\mathbf{R}^{-1} & \mathbf{D} + \mathbf{D}^T \end{bmatrix} \geq 0 \\ &\qquad \mathbf{K} = \mathbf{K}^T \geq 0 \\ &\qquad \begin{bmatrix} t_{p,q} + 2\mathbf{G}_{p,q}^T\mathbf{Q}_{q,q}\widetilde{\mathbf{C}}_{p,q} - \mathbf{G}_{p,q}^T\mathbf{G}_{p,q} & \widetilde{\mathbf{C}}_{p,q} \\ \widetilde{\mathbf{C}}_{p,q}^T & \mathbf{I} \end{bmatrix} \geq 0. \\ &\qquad \forall p, q \end{aligned} \tag{23}$$

[1] Suppose $\mathbf{Q}$ and $\mathbf{R}$ are symmetric. The condition

$$\begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & \mathbf{R} \end{bmatrix} \geq 0 \tag{21}$$

is equivalent to $\mathbf{R} \geq 0$, $\mathbf{Q} - \mathbf{S}\mathbf{R}^{\dagger}\mathbf{S}^T \geq 0$ and $\mathbf{S}(\mathbf{I} - \mathbf{R}\mathbf{R}^{\dagger}) = 0$. Where $\mathbf{R}^{\dagger}$ is the Moore-Penrose inverse of $\mathbf{R}$. We will assume that $\mathbf{R}^{-1}$ exists therefore the linear matrix inequality is equivalent to $\mathbf{R} \geq 0$ and $\mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T \geq 0$.

Note that linear matrix inequality solvers can take advantage of the structure of (10). Also, the passivity constraint may be replaced by a, more general, dissipativity constraint by adding $-2\mathbf{I}\eta$ to the $\mathbf{D}^T + \mathbf{D}$ term [6].

4.1 Experimental Computational Cost

Extensive numerical simulations were conducted to estimate the experimental computational complexity associated with the solution of the constrained fitting problem [9]. Results regarding several scalar problems and two multiport problems are presented in Figure 4.1. In this figure, the multiport model order was considered to be nm where m is the order of each of its n by 1 SIMO subsystems. These results were obtained by formulating the problem in `lmitool` and solving it with the SDP package [1], similar results were obtained with the other packages in `lmitool`.

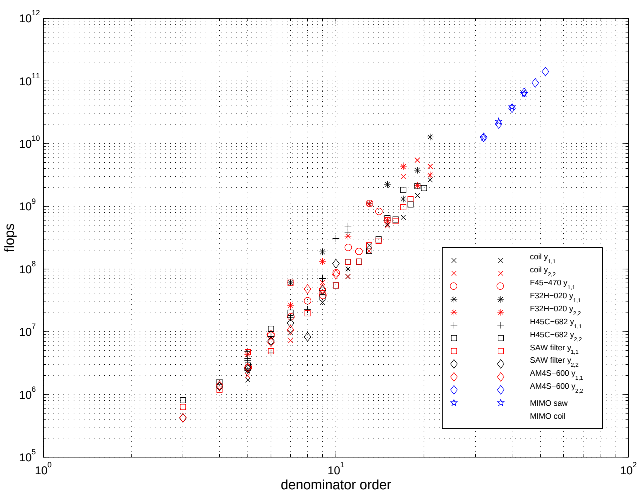

Figure 1: Number of floating point operations associated with solving (23) with the SDPPack package [1].

According to the results in Figure 4.1, the number of floating point operations grows with $(nm)^\alpha$ with $\alpha \in [4.5, 5.5]$. During the course of these numerical experiments it was made clear that, although the problem formulation is quite straightforward, and the LMI problem solving algorithms have a theoretical polynomial worst case upper bound behavior, the direct use of (23), is limited to relatively small problems. The scalability issue, caused by the almost explosive growth of the computational cost with model order, is due to the quadratic growth of the number of unknowns that are the entries of $\mathbf{K}$.

5 INCREMENTAL FITTING STRATEGY

In this section, a modified fitting algorithm that reduces the quadratic increase of the number of variables with the number of ports is presented. It is proposed that the fitting of a multi-port system be split in a series of smaller problems. This divide and conquer strategy does not prove to be optimal but it is believed that the tradeoff between computational cost and model quality may be advantageous in some situations.

The algorithm is based on the premise that, if a matrix function is positive real its minors are also positive real matrix functions. It is proposed that the fitting process start by approximating all the diagonal elements of the rational function matrix. The resulting numerators, $\mathbf{C}_{p,p}$, and $\mathbf{K}_{p,p}$ matrices are then used to determine the first super and sub-diagonal elements by solving (23) for each order two minor on the diagonal. The algorithm then proceeds to determine the second super and sub diagonals by solving (23) for each order three minor on the diagonal. The order of the minors is increased until all the numerators in the matrix rational function are determined. After the n scalar fits of the diagonal elements, each of the remaining $n(n-1)/2$ fitting problems determines two numerator vectors and one $\mathbf{K}_{p,q}$ matrix.

Algorithm 1 Incremental Matrix Fitting Algorithm

for $q = 1$ to n **do**
  determine $\mathbf{C}_{q,q}$ and $\mathbf{K}_{q,q}$ using (23).
end for
for $l = 1$ to n **do**
  for $q = 1$ to $n - l$ **do**
    determine $\mathbf{C}_{q,q+l}$, $\mathbf{C}_{q+l,q}$ and $\mathbf{K}_{q,q+l} = \mathbf{K}^T_{q+l,q}$ by applying (23) to the minor $(q : q + l, q : q + l)$ while considering the, already calculated, $\mathbf{C}_{i,j}$ and $\mathbf{K}_{i,j}$ constant.
  end for
end for

In the previous section, it was determined that the computational cost of solving (23) is given by a polynomial function of the model order nm. Since the number of unknowns depends quadratically on the model order $n_{unknowns} = n^2(m(m+1)/2 + m)$, the computational cost of solving (23) is a polynomial function of $n_{unknowns}$. Moreover, if the computational cost associated with solving (23) is asymptotically proportional to $(nm)^\alpha$ with $\alpha \in [4.5, 5.5]$, it is also asymptotically proportional to $n_{unknowns}^{\alpha-2}$.

Using the proposed scheme, fitting an order m system with n costs $\beta_1 n(m^2 + m)^\alpha + 0.5\beta_2 n(n-1)(m^2 + m)^\alpha$, where β_1 and β_2 are positive constant terms to account for the different cost between the scalar problem and the 2 by 2 restricted problem. It is clear that, asymptotically, for $\alpha > 1$, the proposed scheme is a suboptimal, less expensive, alternative to solving (23).

Numerical experimentations were conducted to determine the effect of the incremental calculation strategy on the computational cost and the error function value. The experiments were carried out for a set of 2-port systems, an almost constant gain in performance by a factor of 5 was observed. For systems with a larger number of ports, this ratio increases and Algorithm 1 becomes more competitive.

It is expected that the incremental fitting algorithm should provide a more accurate fit for the diagonal elements of the admittance matrix. It is also expected that solving the full multivariable problem should result in a better fit of the off-diagonal elements of the admittance matrix.

6 NUMERICAL ISSUES

It was noticed that the positive real lemma linear matrix inequality constraint in (23), was sometimes violated by the existence of very small negative eigenvalues. These numerical error related eigen-

	t_{11}	t_{12}	flops	order
H45C-682-i	1.56×10^0	1.10×10^0	3.26×10^8	2×9
H45C-682-f	1.56×10^0	1.10×10^0	1.86×10^9	2×9
coil-i	1.69×10^0	4.78×10^0	1.33×10^9	2×12
coil-f	2.50×10^0	3.58×10^0	2.49×10^{10}	2×12
SAW-i	9.37×10^{-3}	8.40×10^{-5}	6.95×10^9	2×20
SAW-f	9.37×10^{-3}	8.40×10^{-5}	3.68×10^{10}	2×20

Table 1: Incremental (x-i) and full (x-f) matrix approximation results.

values depend on the semidefinite programming algorithm and its settings.

To eliminate these negative eigenvalues, the following eigenvalue shifted positive real linear matrix inequality was proposed

$$\begin{bmatrix} -\mathbf{A}^T\mathbf{K}-\mathbf{K}\mathbf{A} & -\mathbf{K}\mathbf{B}+\mathbf{C}^T \\ -\mathbf{B}^T\mathbf{K}+\mathbf{C} & \mathbf{D}+\mathbf{D}^T \end{bmatrix} - \eta\mathbf{I} \geq 0. \quad (24)$$

In (24), η is chosen to be a small positive real number greater than the modulus of the most negative eigenvalue. If no negative eigenvalues occur, η may be chosen to be zero. Otherwise it should be increased until a satisfactory solution is obtained. In most cases, if no convergence problems occur, the eigenvalue shift of the affine matrix inequality solves the negative eigenvalue problem without significantly affecting the generated model.

In some cases, the eigenvalue shift, even with a small η has a negative impact on the convergence of the optimization algorithms. In Algorithm 1, the eigenvalue shift η must be smaller than the smallest eigenvalue of $-\mathbf{A}_{q,q}\mathbf{K}_{q,q} - \mathbf{K}_{q,q}\mathbf{A}_{q,q}$ for all q. If this condition is not satisfied, the problem is not feasible.

If the eigenvalue shift method fails, a positive real model may be obtained by setting $\widetilde{\mathbf{D}} = \mathbf{D} + \mathbf{I}\widetilde{\eta}$. The value of $\widetilde{\eta}$, can be increased until the model transfer function is positive real. Since this obviously increases the modeling error, $\widetilde{\eta}$ should be chosen to be as small as possible. Fortunately, in many cases, $\widetilde{\eta}$ is very small.

7 RESULTS

In this section, models for three multivariable two-port passive systems are generated. The pole estimates, required by the proposed algorithms, were obtained by applying the multivariable iterative scaling algorithm to the 2 by 1 admittance vectors that correspond to the SIMO subsystem entries of the $\mathbf{Y}$ parameter matrix. To avoid having some of the entries of the transfer matrix dominating the minimization process, the weights in (15) were chosen to be $w_{k,p,q} = 1/|\widetilde{\mathbf{H}}_{p,q}(s_k)|$. Quadratic error values, $t_{1,1}$ and $t_{1,2}$, and floating point operation count are presented in Table 1. The cost of fitting the scalar diagonal entries of the admittance matrix is included in the cost of Algorithm 1. Due to space restrictions, the full admittance matrices were not displayed. For each example, bode magnitude and phase plots for a port admittance entry (left) and a transfer function (right) entry will be shown. For more results and a detailed description, please see [9].

In the first example, a model for the admittance matrix of a H45C-682 3 port filter from TDK is generated. A 9^{th} order model for each of the columns of the admittance matrix was generated using the multivariable iterative scaling algorithm. A model for each of the scalar diagonal entries of the admittance matrix was generated by solving (23) using an eigenvalue shift of 10^{-10}. The off-diagonal entries, $y_{1,2}$ and $y_{2,1}$, were then approximated by using Algorithm 1. A second model for the admittance matrix was obtained by solving (23). Both multivariable models exhibited very small negative eigenvalues. Due to convergence problems, the eigenvalue shift method did not succeed in eliminating these spurious negative eigenvalues. However, positive real models were obtained by setting $\widetilde{\mathbf{D}}$ to $10^{-16}\mathbf{I}$. This perturbation does not affect the fit. The frequency response of the generated models and the original data are illustrated in Figure 2.

In the next example, a model for the admittance matrix of a coil inductor is generated. A 12^{th} order model for each of the columns of the admittance matrix was generated using the multivariable iterative scaling algorithm. A model for each of the scalar diagonal entries of the admittance matrix was generated by solving (23). The off-diagonal entries, $y_{1,2}$ and $y_{2,1}$, were then approximated by using Algorithm 1. A second model for the admittance matrix was obtained by solving (23) using an eigenvalue shift of 10^{-10}. The frequency response of the models and the original data are illustrated in Figure 3.

In our final example, a model for the admittance matrix of a SAW filter is generated. A 20^{th} order model for each of the columns of the admittance matrix was generated using the multivariable iterative scaling algorithm. The $\mathbf{D}$ matrix was arbitrated to be $\mathbf{I} \times 10^{-5}$. A model for each of the scalar diagonal entries of the admittance matrix was generated by solving (23). The off-diagonal entries, $y_{1,2}$ and $y_{2,1}$, were then approximated by using Algorithm 1. A second model for the admittance matrix was obtained by solving (23). In all cases, the weighting coefficients were chosen to be $w_{k,p,q} = 1/|\mathbf{H}_{p,q}(s_k)|$ for all p, q and k. No eigenvalue shift was necessary. The resulting models satisfy (16). The frequency response of the models and the original data are illustrated in Figure 4.

8 CONCLUSIONS

In this paper, the fixed denominator positive real constrained rational approximation problem was formulated as a linear program with linear matrix inequality constraints. Since this is a convex optimization problem, a globally optimal solution may be found using interior point methods such as those proposed by Nesterov and Nemirosvskii [6].

The direct use of the positive real lemma and an adequate representation of the optimization variables allowed for a straightforward problem formulation for both scalar and multivariable systems. Unfortunately, it also introduces a large amount of redundancy which severely limits the scalability of the proposed problem formulation. In order to reduce the cost of modeling multivariable systems, an incremental fitting strategy was proposed and tested.

It was noticed that, in certain cases, the algorithms used to solve the proposed optimization problems failed either by violating the positive definiteness of the matrix inequalities or by failing to converge. To correct the positive definiteness violations, an eigenvalue shifted problem formulation was proposed.

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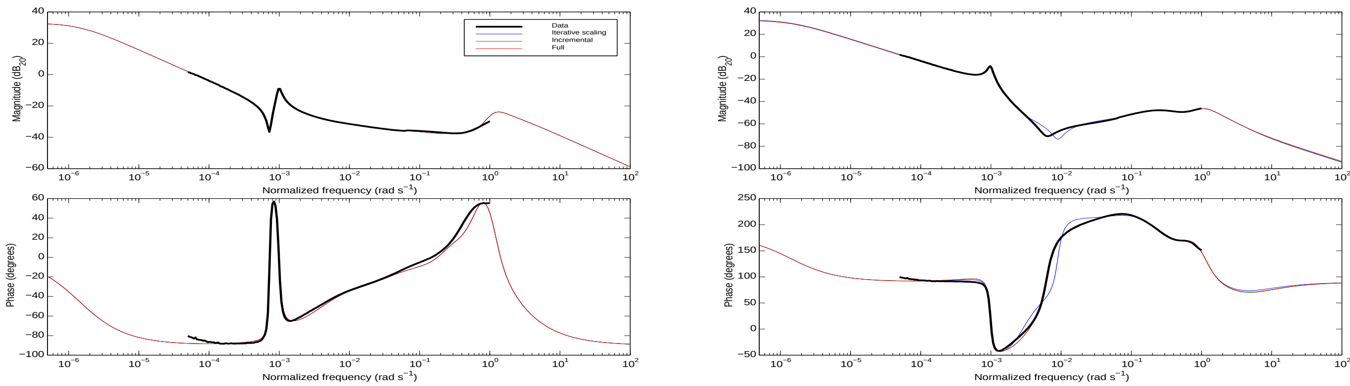

Figure 2: Multivariable passive constrained fit of the admittance matrix of a H45C-682 3-port filter from TDK.

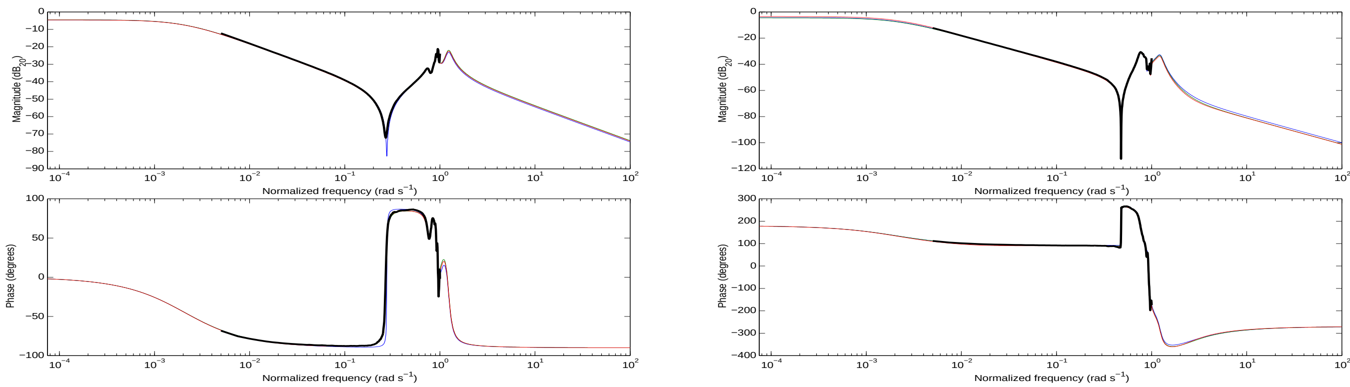

Figure 3: Multivariable passive constrained fit of the admittance matrix of a coil inductor.

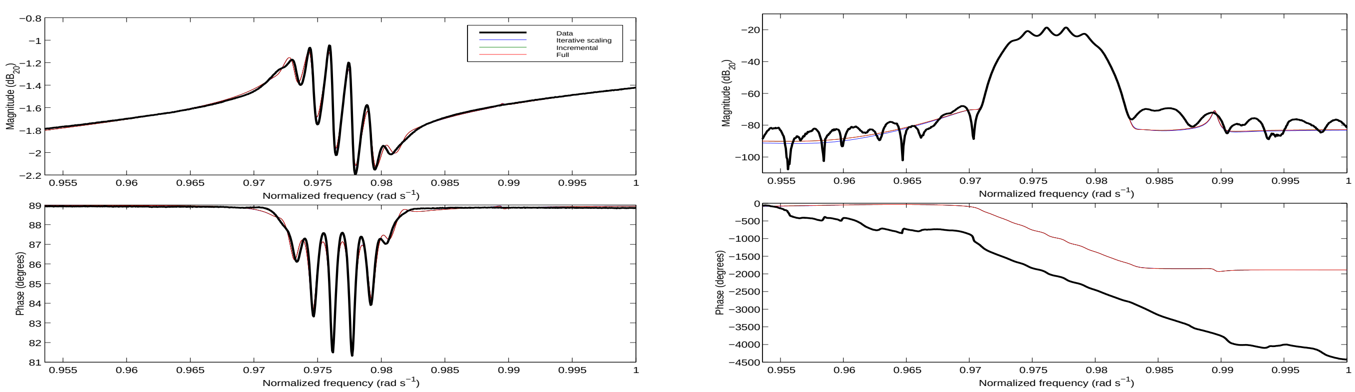

Figure 4: Multivariable passive constrained fit of the admittance matrix of a SAW filter.

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