Sampling and Reconstruction in Arbitrary Measurement and Approximation Spaces Associated with Linear Canonical Transform

Jun Shi, Xiaoping Liu, Lei He, Mo Han, Qingzhong Li, and Naitong Zhang

***Abstract*—The linear canonical transform (LCT), which generalizes many classical transforms, has been shown to be a powerful tool for signal processing and optics. Sampling theory of the LCT for bandlimited signals has blossomed in recent years. However, in practice signals are never perfectly bandlimited, and in many cases measurement devices are non-ideal. The objective of this paper is to develop a sampling theorem for the LCT from general measurements, which can provide a suitable and realistic model of sampling and approximation for real-world applications. We first describe a general class of approximation spaces for the LCT and provide a full characterization of their basis functions. Then, we propose a generalized sampling theorem for arbitrary measurement and approximation spaces associated with the LCT. Several properties of the proposed sampling theorem are also discussed. Furthermore, the approximation error is estimated. Finally, numerical results and several applications of the derived results are presented.**

***Index Terms*—Linear canonical transform, Riesz basis, oblique projection, function spaces, sampling and approximation.**

I. INTRODUCTION

THE linear canonical transform (LCT) [1]–[3] has recently attracted much attention as a powerful mathematical tool for signal and system analysis in signal processing and optics. It forms a three-parameter family of integral transforms and was first introduced in the 1970s [4], [5]. The LCT is also known as the ABCD transform, the affine Fourier transform, the Collins formula, the generalized Fresnel transform, or the almost Fourier and almost Fresnel transformation [3]. It has found applications in fields as diverse as filter design, radar system analysis, signal synthesis, time-frequency analysis, phase retrieval, pattern recognition, graded index media analysis, encryption, and communications [1]–[18].

The LCT of a function $f(t) \in L^2(\mathbb{R})$ is defined as [2]

$$\begin{aligned} F_M(u) &= \mathcal{L}^M\{f(t)\}(u) \\ &= \begin{cases} \int_{\mathbb{R}} f(t) K_M(u,t)dt, & b \neq 0 \\ \sqrt{d} e^{\frac{jcd}{2}u^2} f(du), & b = 0 \end{cases} \end{aligned} \tag{1}$$

where $\mathcal{L}^M$ denotes the LCT operator, and the transformation kernel is given by

$$K_M(u,t) = A_b e^{\frac{ja}{2b}t^2 + \frac{jd}{2b}u^2 - j\frac{u}{b}t} \tag{2}$$

where $A_b = \frac{1}{\sqrt{j2\pi b}}$, and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the parameter matrix of LCT satisfying $\det(M) = ad - bc = 1$. Note that when $b = 0$, the LCT is essentially a chirp multiplication operation. Therefore, from now on we consider the LCT for $b \neq 0$, and without loss of generality, we assume $b > 0$. The LCT parametrically generalizes a number of well-known unitary transforms linked with signal processing and optics. For $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the LCT amounts to the classical Fourier transform (FT). Similarly, with $M = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$, one can obtain the fractional Fourier transform (FRFT). The Fresnel, Laplace, Gauss-Weierstrass, and Bargmann transforms are also particular cases [3]. Conversely, the inverse LCT with respect to matrix M is the LCT with matrix $M^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$, i.e.,

$$f(t) = \mathcal{L}^{M^{-1}}\{F_M(u)\}(t) = \int_{\mathbb{R}} F_M(u) K_M^*(u,t)du \tag{3}$$

where $*$ in the superscript denotes the complex conjugate.

Signal processing applications are concerned mainly with digital data, although the origin of many sources of information is analog [19]. Therefore, one of the most fundamental issues in signal processing associated with the LCT is how to represent a continuous signal in terms of a discrete sequence. The most common setting considered in the sampling theory of the LCT [20]–[28] follows the same paradigm as the classical Shannon sampling theorem [29], [30], in which the input is assumed to be bandlimited in the LCT domain, the samples of the signals are ideal, i.e., they are equal to the signal values at a set of sampling points, and the reconstructed signal is also an LCT-bandlimited function, created by using the sinc interpolation kernel, i.e.,

$$\tilde{f}(t) = \sum_{n\in\mathbb{Z}} f[n] \operatorname{sinc}(t-n) e^{-\frac{ja}{2b}(t^2-n^2)} \tag{4}$$

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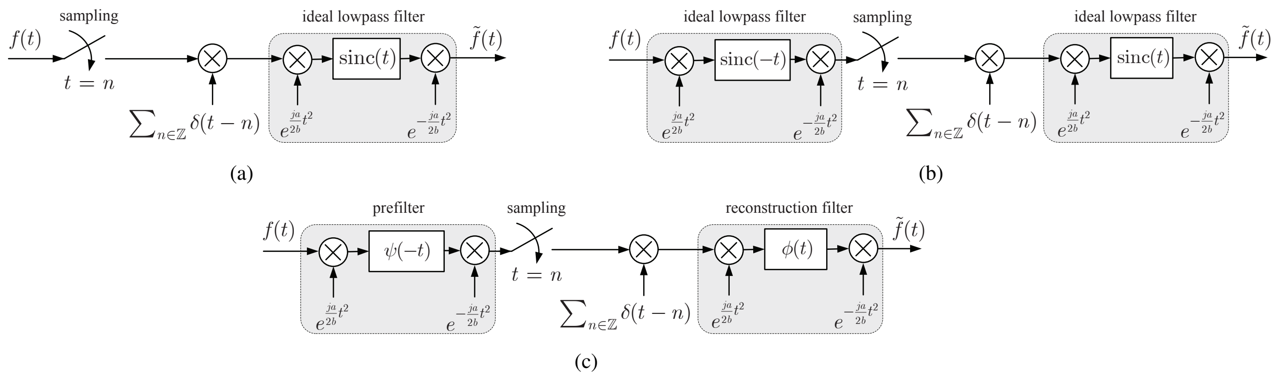

Fig. 1. Block diagram representation of three sampling procedures associated with the LCT: (a) bandlimited interpolation, (b) bandlimited interpolation with an ideal anti-aliasing lowpass prefilter, and (c) least squares approximation.

where $\mathrm{sinc}(\cdot) \triangleq \sin\pi(\cdot)/\pi(\cdot)$, and a normalized sampling step is used to keep the notation simple. The reconstructed signal $\tilde{f}(t)$ can be obtained by modulating the sequence $\{f[n]\}_{n\in\mathbb{Z}}$ with an impulse train signal $\sum_{n\in\mathbb{Z}} \delta(t-n)$, and then by filtering the modulated signal in the LCT domain using a filter with impulse response $\mathrm{sinc}(t)$, as depicted in Fig. 1(a). However, real world signals or images are never exactly bandlimited, and the sampling is never ideal. Towards this end, Liu *et al.* [31] proposed new sampling expansions for non-bandlimited signals by introducing certain types of non-bandlimited function spaces associated with the LCT. Unfortunately, as the authors of [31] pointed out, there are no normative rules for determining the parameters of non-bandlimited function spaces in practical implementations at present. Within Shannon's paradigm for the LCT, when the input signal $f(t)$ is not bandlimited, it needs to be ideally lowpass filtered prior to sampling in order to avoid aliasing, see Fig. 1(b). Clearly, there is no such device as an ideal (anti-aliasing or reconstruction) low-pass filter, and the drawback of the sampling procedure in Fig. 1(b) is the difficulty in implementing the infinite sinc interpolating kernel, which has slow decay. Mathematically, the bandlimited sampling procedure in the LCT domain is equivalent to projecting the input signal $f(t)$ onto the LCT-bandlimited function space [32]

$$\mathcal{B}_M = \overline{\mathrm{span}}\left\{\mathrm{sinc}(t-n)e^{-\frac{ja}{2b}(t^2-n^2)}\right\}_{n\in\mathbb{Z}}. \tag{5}$$

It was shown in [32] that one can determine the optimal (least squares) approximation of the input signal $f(t)$ in some more general function space

$$\mathcal{V}_M(\phi) = \overline{\mathrm{span}}\left\{\phi(t-n)e^{-\frac{ja}{2b}(t^2-n^2)}\right\}_{n\in\mathbb{Z}} \tag{6}$$

where $\phi(t)$ is the generating function of the space and can basically be arbitrary, as illustrated in Fig. 1(c). The optimal setting for this general case is to choose a measurement function $\psi(t)$ that is biorthogonal to $\phi(t)$. However, in most applications, the measurement function is often specified *a priori* [30] and corresponds to the impulse response of the measurement device, denoted by $\chi(t)$. From [32], the approximation error is minimized if and only if $\chi(t) = \psi(-t)$. This situation corresponds to the case of an ideal measurement device. If this condition is not satisfied, different types of errors (distortion, aliasing) will be introduced into the sampling process, resulting in a significant loss of performance. In this paper, we will introduce a general framework for sampling and approximation in the LCT domain, which can provide a suitable and realistic model of digital signal processing for real-world applications. The sampling scheme which we develop allows for almost arbitrary measurement and approximation spaces, as well as arbitrary input signals. We first describe the general class of approximation spaces associated with the LCT and provide a full characterization of their basis functions. Then, we propose a generalized sampling theorem in arbitrary measurement and approximation spaces associated with the LCT. Several properties of the proposed sampling theorem are also discussed. Furthermore, the analysis of approximation error is derived. Numerical results and several applications of the derived results are presented.

The remainder of this paper is organized as follows. In Section II, notation and definitions are introduced, and some facts of the discrete-time LCT and approximation spaces associated with the LCT are given. In Section III, a sampling theorem for the LCT from general measurements is developed, and its basic properties are also presented. In Section IV, a bound on the approximation error for the proposed sampling theorem is derived. In Section V, numerical results and several applications of the derived results are presented. Finally, concluding remarks are drawn in Section VI.

II. Preliminaries

A. Notation and Definitions

We use the following notation throughout: $\mathbb{R}$, $\mathbb{Z}$, $L^2(\mathbb{R})$, $L^2[a,b]$, and $\ell^2(\mathbb{Z})$ denote the set of real numbers, the set of integers, the space of all square-integrable functions on $\mathbb{R}$, the space of all square-integrable functions on $[a,b]$, and the space of all square-summable sequences on $\mathbb{Z}$, respectively. Continuous signals are denoted with parentheses, e.g., $f(t)$, $t \in \mathbb{R}$, and discrete signals with brackets, e.g., $q[n]$, $n \in \mathbb{Z}$. We denote the L^2-inner product between $f(t)$ and $g(t)$ by $\langle f, g\rangle_{L^2} = \int_{\mathbb{R}} f(t)g^*(t)dt$, and the ℓ^2-inner product between $p[n]$ and $q[n]$ by $\langle p, q\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} p[n]q^*[n]$ so that the squared L^2- and ℓ^2-norms can be expressed as $\|\cdot\|^2_{L^2} = \langle\cdot,\cdot\rangle_{L^2}$ and $\|\cdot\|^2_{\ell^2} = \langle\cdot,\cdot\rangle_{\ell^2}$, respectively. For a measurable function $f(t)$ on $\mathbb{R}$, let $\|f\|_0 = \operatorname{ess\,inf} |f(t)|$ be the infimum of $|f(t)|$. Unless otherwise stated, $\mathcal{H}$ will always denote an arbitrary (real or complex) Hilbert space. By a subspace of $\mathcal{H}$, we will always mean a linear subspace.

The minimal angle $\theta_{\min} \in [0, \pi/2]$ between two nontrivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ is defined as [33]

$$S(\mathcal{M},\mathcal{N}) = \cos(\theta_{\min}) = \sup_{\substack{f\in\mathcal{M}\setminus\{0\}\\ g\in\mathcal{N}\setminus\{0\}}} \left\{ \frac{\langle f, g\rangle_{L^2}}{\|f\|_{L^2}\cdot\|g\|_{L^2}} \right\} \tag{7}$$

which clearly shows that $S(\mathcal{M},\mathcal{N}) = S(\mathcal{N},\mathcal{M})$. Taking into account the fact that for all $z_0(t) \in \mathcal{N}$ we have

$$\|z_0\|_{L^2} = \sup_{f\in\mathcal{N}\setminus\{0\}} \left\{ \frac{\langle z_0, f\rangle_{L^2}}{\|f\|_{L^2}} \right\}, \tag{8}$$

the expression in (7) can be rewritten as

$$\begin{aligned} S(\mathcal{M},\mathcal{N}) &= \sup_{\substack{f\in\mathcal{M}\setminus\{0\}\\ g\in\mathcal{N}\setminus\{0\}}} \left\{ \frac{\langle \mathcal{P}_{\mathcal{N}} f, g\rangle_{L^2}}{\|f\|_{L^2}\cdot\|g\|_{L^2}} \right\} \\ &= \sup_{f\in\mathcal{M}\setminus\{0\}} \left\{ \frac{\|\mathcal{P}_{\mathcal{N}} f\|_{L^2}}{\|f\|_{L^2}} \right\} \end{aligned} \tag{9}$$

where $\mathcal{P}_{\mathcal{N}}$ is the orthogonal projection onto $\mathcal{N}$. Similarly, the maximal angle $\theta_{\max} \in [0, \pi/2]$ between two nontrivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is defined as

$$R(\mathcal{M},\mathcal{N}) = \cos(\theta_{\max}) = \inf_{f\in\mathcal{M}\setminus\{0\}} \left\{ \frac{\|\mathcal{P}_{\mathcal{N}} f\|_{L^2}}{\|f\|_{L^2}} \right\}. \tag{10}$$

In general, $R(\mathcal{M},\mathcal{N})$ and $R(\mathcal{N},\mathcal{M})$ are not necessarily equal, but we always have

$$R(\mathcal{M},\mathcal{N}) = R(\mathcal{M}^{\perp},\mathcal{N}^{\perp}). \tag{11}$$

The relationship between the minimal and maximal angles is expressed as

$$S^2(\mathcal{M},\mathcal{N}) + R^2(\mathcal{M},\mathcal{N}^{\perp}) = 1. \tag{12}$$

B. Discrete-time LCT and Its Properties

The discrete-time LCT (DTLCT) of a sequence $\{s[n]\}_{n\in\mathbb{Z}}$ is defined as [34]

$$\widetilde{S}_M(u) = \widetilde{\mathcal{L}}^M\{s[n]\}(u) = \sum_{n\in\mathbb{Z}} s[n] K_M(u,n) \tag{13}$$

where $\widetilde{\mathcal{L}}^M$ denotes the DTLCT operator. It is easy to see that if the sequence $\{s[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then $\widetilde{S}_M(u) \in L^2(\mathbb{I})$, where $\mathbb{I} \triangleq \mathbb{I}$. Conversely, the inverse DTLCT is given by

$$s[n] = \int_{\mathbb{I}} \widetilde{S}_M(u) K_M^*(u,n) du. \tag{14}$$

The Parseval identity of the DTLCT is expressed as

$$\|s\|_{\ell^2}^2 = \int_{\mathbb{I}} \left|\widetilde{S}_M(u)\right|^2 du. \tag{15}$$

Moreover, the DTLCT has the following chirp-periodicity

$$\widetilde{S}_M(u + 2k\pi b)\, e^{-\frac{jd}{2b}(u+2k\pi b)^2} = \widetilde{S}_M(u) e^{-\frac{jd}{2b}u^2} \tag{16}$$

for any $k \in \mathbb{Z}$.

It was shown in [18] that there are several definitions for canonical convolution of the LCT in the literature. The relationships among them were investigated in [18] in detail. We use the one introduced in [32], which inherits the structure of the Shannon reconstruction formula for the LCT shown in (4). Let Θ_{f} denote the fully-discrete canonical convolution operator associated with the DTLCT. The canonical convolution of $\{s[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $\{h[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is defined as [32]

$$s[n]\Theta_{\mathrm{f}} h[n] = \sum_{m\in\mathbb{Z}} s[m] h[n-m] e^{-\frac{ja}{2b}(n^2-m^2)}. \tag{17}$$

It is easy to verify that

$$s[n]\Theta_{\mathrm{f}} h[n] \xleftrightarrow{\widetilde{\mathcal{L}}^M} \sqrt{2\pi}\,\widetilde{S}_M(u)\widetilde{H}\left(\tfrac{u}{b}\right) \tag{18}$$

where $\widetilde{H}\left(\frac{u}{b}\right)$ denotes the discrete-time FT (DTFT) (with its argument scaled by $\frac{1}{b}$) of $h[n]$. Let $g(t) \in L^2(\mathbb{R})$. The semi-discrete canonical convolution operator Θ_{s} of the LCT is defined as a linear map from $\ell^2(\mathbb{Z})$ into $L^2(\mathbb{R})$ such that [32]

$$s[n]\Theta_{\mathrm{s}} g(t) = \sum_{n\in\mathbb{Z}} s[n] g(t-n) e^{-\frac{ja}{2b}(t^2-n^2)} \tag{19}$$

which satisfies

$$s[n]\Theta_{\mathrm{s}} g(t) \xleftrightarrow{\mathcal{L}^M} \sqrt{2\pi}\,\widetilde{S}_M(u) G\left(\tfrac{u}{b}\right) \tag{20}$$

where $G\left(\frac{u}{b}\right)$ denotes the FT (with its argument scaled by $\frac{1}{b}$) of $g(t)$.

C. Approximation Spaces Associated with LCT

For a function $\phi(t) \in L^2(\mathbb{R})$, the signal approximation space $\mathcal{V}_M(\phi)$ in (6) can be rewritten as

$$\mathcal{V}_M(\phi) = \left\{ \sum_{n\in\mathbb{Z}} c[n]\phi_{n,M}(t) \middle| c[n] \in \ell^2(\mathbb{Z}) \right\} \tag{21}$$

where $\phi_{n,M}(t)$ is defined as

$$\phi_{n,M}(t) \triangleq \phi(t-n) e^{-\frac{ja}{2b}(t^2-n^2)}. \tag{22}$$

In general, the function sequence $\{\phi_{n,M}(t)\}_{n\in\mathbb{Z}}$ is not a Riesz basis of $\mathcal{V}_M(\phi)$ [35]. In fact, it is a Riesz basis of $\mathcal{V}_M(\phi)$ if and only if there exist two positive constants A and B such that [32]

$$A \le \sum_{k\in\mathbb{Z}} \left|\Phi\left(\tfrac{u}{b} + 2k\pi\right)\right|^2 \le B \tag{23}$$

where $\Phi\left(\frac{u}{b}\right)$ denotes the FT (with its argument scaled by $\frac{1}{b}$) of $\phi(t)$. The equality holds if and only if $\{\phi_{n,M}(t)\}_{n\in\mathbb{Z}}$ is orthonormal, i.e., when $A = B = 1$.

Note that $\phi(t)$ is not the only function that can generate the space $\mathcal{V}_M(\phi)$. By following the steps in [32] and [36], it is possible to construct many others by using general equivalent generating function $\phi_{\mathrm{eq}}(t) \in L^2(\mathbb{R})$ of the form

$$\begin{aligned} \phi_{\mathrm{eq}}(t) e^{-\frac{ja}{2b}t^2} &= \sum_{n\in\mathbb{Z}} q[n]\phi_{n,M}(t) \\ &= q[n]\Theta_{\mathrm{s}}\phi(t) \end{aligned} \tag{24}$$

where $q[n]$ is an appropriate sequence of weights. The necessary and sufficient condition for the function sequence

TABLE I
PRIMARY TYPES OF EQUIVALENT GENERATING FUNCTIONS WITH THEIR SPECIFIC PROPERTIES

Type	The DTLCT of $q[n]$	Property
Generic	$\widetilde{Q}_M(u) = \frac{A_b e^{\frac{jd}{2b}u^2}}{\sqrt{2\pi}}$	Admissibility: $0 < A \le \sum_{k\in\mathbb{Z}} \left|\Phi_{\text{eq}}\left(\frac{u}{b}+2k\pi\right)\right|^2 \le B$
Interpolating	$\widetilde{Q}_M(u) = \frac{A_b e^{\frac{jd}{2b}u^2}}{\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\Phi\left(\frac{u}{b}+2k\pi\right)}$	Interpolating: $\phi_{\text{eq}}(k) = \delta_k,\ k\in\mathbb{Z}$
Orthogonal	$\widetilde{Q}_M(u) = \frac{A_b e^{\frac{jd}{2b}u^2}}{\sqrt{2\pi}\sqrt{\sum_{k\in\mathbb{Z}}\left|\Phi\left(\frac{u}{b}+2k\pi\right)\right|^2}}$	Orthogonality: $\langle \phi_{\text{eq},n,M}(t), \phi_{\text{eq},k,M}(t)\rangle = \delta_{k-n}$
Dual	$\widetilde{Q}_M(u) = \frac{A_b e^{\frac{jd}{2b}u^2}}{\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\left|\Phi\left(\frac{u}{b}+2k\pi\right)\right|^2}$	Biorthogonality: $\langle \phi_{\text{eq},n,M}(t), \phi_{k,M}(t)\rangle = \delta_{k-n}$

$\left\{\phi_{\text{eq},n,M}(t) \triangleq \phi_{\text{eq}}(t-n)e^{-\frac{ja}{2b}(t^2-n^2)}\right\}_{n\in\mathbb{Z}}$ to yield an equivalent Riesz basis of $\mathcal{V}_M(\phi)$ is that there exist two strictly positive constants ϱ_0 and ϱ_1 such that

$$\varrho_0 \le |\widetilde{Q}_M(u)| \le \varrho_1 \quad \text{a.e.} \tag{25}$$

where $\widetilde{Q}_M(u)$ denotes the DTLCT of $q[n]$. In fact, the sequence $q[n]$ can be chosen so that the basis functions satisfy certain prescribed properties. The most important types of generating functions are summarized in Table I, where $\Phi_{\text{eq}}(\frac{u}{b})$ denotes the FT (with its argument scaled by $\frac{1}{b}$) of $\phi_{\text{eq}}(t)$. For each set of basis functions, there exists a unique set of coefficients that characterizes the function in $\mathcal{V}_M(\phi)$. The selection of the most appropriate representation generally depends on the application.

III. SAMPLING AND RECONSTRUCTION IN ARBITRARY MEASUREMENT AND APPROXIMATION SPACES ASSOCIATED WITH LCT

Many methods exist for representing a signal $f(t)$ by a sequence of numbers, which can be interpreted as measurements of $f(t)$. The ordinary approach is to choose the measurements as samples of $f(t)$. Here, we consider a generalized sampling scheme as shown in Fig. 2, in which the measurements can be expressed as inner products of $f(t)$ with a function sequence $\{\upsilon_{n,M}(t) \triangleq \upsilon(t-n)e^{-\frac{ja}{2b}(t^2-n^2)}\}_{n\in\mathbb{Z}}$ that spans a subspace $\mathcal{S}_M(\upsilon)$, which is referred to as the measurement space. The problem then is to reconstruct $f(t)$ from these measurements by using a function sequence $\{\phi_{n,M}(t)\}_{n\in\mathbb{Z}}$ that spans the approximation space $\mathcal{V}_M(\phi)$.

A. Generalized Sampling Theorem for LCT

In Fig. 1(c), the orthogonal projection of $f(t) \in L^2(\mathbb{R})$ on $\mathcal{V}_M(\phi)$ provides the optimal representation in the sense that the L^2-approximation error is minimized. This least squares solution can be computed simply by prefiltering and sampling [32]. The corresponding optimal prefilter $\psi(t)$ with $\psi(t)e^{-j\frac{ja}{2b}t^2} \in \mathcal{V}_M(\phi)$ is uniquely specified and corresponds to the dual of the generating function $\phi(t)$ of $\mathcal{V}_M(\phi)$. This implies that the least squares solution to the generalized sampling system of Fig. 2 requires the measurement space $\mathcal{S}_M(\upsilon)$ to be equal to the approximation space $\mathcal{V}_M(\phi)$. In practice, the prefilter typically corresponds to the impulse response of the measurement device. In most cases, this operator is specified *a priori*, and thus it is usually not possible to obtain the least squares solution directly. Therefore, we must relax the requirement for perfect reconstruction. Note in Fig. 2, it will acquire uniformly spaced samples at the output of a measurement device to produce an approximation $\tilde{f}(t) \in \mathcal{V}_M(\phi)$ of the input function $f(t) \in L^2(\mathbb{R})$. The function $\upsilon(-t)$ is the impulse response of the measurement device, and the sequence $\{h[n]\}_{n\in\mathbb{Z}}$ is a digital correction filter to be specified. Both $\mathcal{S}_M(\upsilon)$ and $\mathcal{V}_M(\phi)$ are defined by an equation equivalent to (21). In addition, the generating functions $\upsilon(t)$ and $\phi(t)$ both satisfy the Riesz basis condition (23). To specify the signal approximation $\tilde{f}(t)$, we introduce a constraint on our sampling procedure, i.e.,

$$c_1[n] = \langle f(t), \upsilon_{n,M}(t)\rangle_{L^2} = \langle \tilde{f}(t), \upsilon_{n,M}(t)\rangle_{L^2}. \tag{26}$$

For simplicity, we define the sampled cross-correlation sequence of $\phi(t)$ and $\upsilon(t)$ as

$$\lambda_{\phi\upsilon}[k] \triangleq \langle \phi(t), \upsilon(t-k)\rangle_{L^2}. \tag{27}$$

Let $\widetilde{\Lambda}_{\phi\upsilon}(\frac{u}{b})$ denote the DTFT (with its argument scaled by $\frac{1}{b}$) of $\lambda_{\phi\upsilon}[k]$. Taking the corresponding scaled DTFTs of both

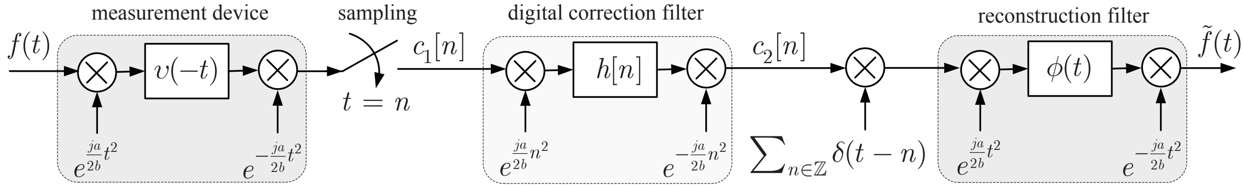

Fig. 2. Sampling in a general measurement space $\mathcal{S}_M(\upsilon)$ with signal reconstruction in the approximation space $\mathcal{V}_M(\phi)$.

sides of (27) results in

$$\widetilde{\Lambda}_{\phi\upsilon}\left(\tfrac{u}{b}\right)=\sqrt{2\pi}\sum_{k\in\mathbb{Z}}\Phi^{*}\left(\tfrac{u}{b}+2k\pi\right)\Upsilon\left(\tfrac{u}{b}+2k\pi\right) \tag{28}$$

where $\Upsilon(\frac{u}{b})$ indicates the FT (with its argument scaled by $\frac{1}{b}$) of $\upsilon(t)$. Based on the above results, we have the following generalized sampling theorem associated with the LCT.

Theorem 1: A unique solution satisfying (26) exists if and only if there are two positive constants ϵ_0 and ϵ_1 such that $\epsilon_0 \le |\widetilde{\Lambda}_{\phi\upsilon}\left(\frac{u}{b}\right)| \le \epsilon_1$. Let $\{h[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ be a sequence with its DTFT $\widetilde{H}\left(\frac{u}{b}\right)=1/(2\pi\widetilde{\Lambda}_{\phi\upsilon}(\frac{u}{b}))$. The solution is the projection of $f(t)$ on $\mathcal{V}_M(\phi)$ perpendicular to $\mathcal{S}_M(\upsilon)$, i.e.,

$$\begin{aligned}\tilde{f}(t)&=\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)\\&=\sum_{n\in\mathbb{Z}}c_2[n]\phi_{n,M}(t)\end{aligned} \tag{29}$$

where expansion coefficients $c_2[n]$ is determined as

$$\begin{aligned}c_2[n]&=(c_1\Theta_{\mathrm{f}}h)[n]\\&=\sum_{k\in\mathbb{Z}}c_1[k]h[n-k]e^{-\frac{ja}{2b}(n^2-k^2)}.\end{aligned} \tag{30}$$

Proof: The function $\tilde{f}(t)$ can be rewritten as

$$\tilde{f}(t)=\sum_{n\in\mathbb{Z}}(c_1\Theta_{\mathrm{f}}h)[n]\phi_{n,M}(t). \tag{31}$$

Inserting (31) into (26) results in

$$\begin{aligned}c_1[n]&=\Big\langle\sum_{m\in\mathbb{Z}}(c_1\Theta_{\mathrm{f}}h)[m]\phi_{m,M}(t),\upsilon_{n,M}(t)\Big\rangle_{L^2}\\&=\sum_{m\in\mathbb{Z}}(c_1\Theta_{\mathrm{f}}h)[m]\left\langle\phi_{m,M}(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\end{aligned} \tag{32}$$

which, in conjunction with (22) and (27), gives rise to

$$\begin{aligned}c_1[n]&=\sum_{m\in\mathbb{Z}}(c_1\Theta_{\mathrm{f}}h)[m]\lambda_{\phi\upsilon}[n-m]e^{-\frac{ja}{2b}(n^2-m^2)}\\&=(c_1\Theta_{\mathrm{f}}h)[n]\Theta_{\mathrm{f}}\lambda_{\phi\upsilon}[n].\end{aligned} \tag{33}$$

Then, from (18) and (33), we have

$$\widetilde{C}_{1,M}(u)=2\pi\widetilde{C}_{1,M}(u)\widetilde{H}\left(\tfrac{u}{b}\right)\widetilde{\Lambda}_{\phi\upsilon}\left(\tfrac{u}{b}\right) \tag{34}$$

where $\widetilde{C}_{1,M}(u)$ denotes the DTLCT of $c_1[n]$. Thus, we derive

$$\widetilde{H}\left(\tfrac{u}{b}\right)=\frac{1}{2\pi\widetilde{\Lambda}_{\phi\upsilon}\left(\frac{u}{b}\right)}. \tag{35}$$

It is not difficult to see that (29) defines a linear operator $\mathcal{P}: L^2(\mathbb{R})\to\mathcal{V}_M(\phi)$. To prove that $\mathcal{P}$ is the projection on $\mathcal{V}_M(\phi)$ perpendicular to $\mathcal{S}_M(\upsilon)$, we will show that $\mathcal{P}$ has the following properties:

$$\forall g(t)\in\mathcal{V}_M(\phi),\quad \mathcal{P}g(t)=g(t) \tag{36}$$

$$\forall f(t)\in L^2(\mathbb{R}),\quad f(t)-\mathcal{P}f(t)\in\mathcal{S}_M^{\perp}(\upsilon) \tag{37}$$

$$\forall e(t)\in\mathcal{S}_M^{\perp}(\upsilon),\quad \mathcal{P}e(t)=0. \tag{38}$$

First, since $g(t)\in\mathcal{V}_M(\phi)$, there exists a scalar sequence $\{c[m]\}_{m\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ such that

$$g(t)=\sum_{m\in\mathbb{Z}}c[m]\phi_{m,M}(t). \tag{39}$$

By combining (39), (26) and (31), the expansion coefficients of $\mathcal{P}g(t)$ can be derived as

$$\begin{aligned}&\left\langle g(t),\upsilon_{n,M}(t)\right\rangle\Theta_{\mathrm{f}}h[n]\\&\quad=\Big\langle\sum_{m\in\mathbb{Z}}c[m]\phi_{m,M}(t),\upsilon_{n,M}(t)\Big\rangle_{L^2}\Theta_{\mathrm{f}}h[n]\\&\quad=\sum_{m\in\mathbb{Z}}c[m]\left\langle\phi_{m,M}(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\Theta_{\mathrm{f}}h[n]\\&\quad=\Big(c[n]\Theta_{\mathrm{f}}\lambda_{\phi\upsilon}[n]\Big)\Theta_{\mathrm{f}}h[n].\end{aligned} \tag{40}$$

Taking the DTLCTs of both sides of (40) and applying (18) and (35) yield

$$\begin{aligned}&\widetilde{\mathcal{L}}^M\left\{\left\langle g(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\Theta_{\mathrm{f}}h[n]\right\}(u)\\&\qquad=2\pi\widetilde{C}_M(u)\widetilde{\Lambda}_{\phi\upsilon}(\tfrac{u}{b})\widetilde{H}(\tfrac{u}{b})\\&\qquad=\widetilde{C}_M(u)\end{aligned} \tag{41}$$

where $\widetilde{C}_M(u)$ denotes the DTLCT of $c[n]$. Thus, we have

$$\left\langle g(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\Theta_{\mathrm{f}}h[n]=c[n] \tag{42}$$

which validates that (36) is true. That is, $\mathcal{P}$ is a projector on $\mathcal{V}_M(\phi)$. Next, we consider the following inner product

$$\begin{aligned}&\left\langle f(t)-\mathcal{P}f(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\\&\quad=\left\langle f(t),\upsilon_{n,M}(t)\right\rangle_{L^2}-\left\langle\mathcal{P}f(t),\upsilon_{n,M}(t)\right\rangle_{L^2}\\&\quad=c_1[n]-\Big\langle\sum_{m\in\mathbb{Z}}c_2[m]\phi_{m,M}(t),\upsilon_{n,M}(t)\Big\rangle_{L^2}\\&\quad=c_1[n]-(c_2\Theta_{\mathrm{f}}\lambda_{\phi\upsilon})[n].\end{aligned} \tag{43}$$

By applying (43), (33), and (30), it follows that

$$\left\langle f(t)-\mathcal{P}f(t),\upsilon_{n,M}(t)\right\rangle_{L^2}=0. \tag{44}$$

This result implies that the projection error is orthogonal to $\mathcal{S}_M(\upsilon)$, i.e., (37) holds. Finally, we can use the fact that

$$\forall e(t)\in\mathcal{S}_M^{\perp}(\upsilon),\quad\left\langle e(t),\upsilon_{n,M}(t)\right\rangle_{L^2}=0 \tag{45}$$

which, along with (26) and (31), implies that (38) is satisfied as well. This completes the proof of Theorem 1. □

B. Discussion and Properties

1) Equivalent Biorthogonal Expansion: The solution of (26) described by Theorem 1 can also be expanded as a set of biorthogonal basis functions, i.e.,

$$\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)=\sum_{n\in\mathbb{Z}}\left\langle f(t),\mathring{\phi}_{n,M}(t)\right\rangle_{L^2}\phi_{n,M}(t) \tag{46}$$

where the equivalent biorthogonal function $\mathring{\phi}_{n,M}(t)$ satisfying $\mathring{\phi}_{n,M}(t)\in\mathcal{S}_M(\upsilon)$ is unique and inherits the structure of $\phi_{n,M}(t)$ in (22), satisfying

$$\mathring{\Phi}\left(\tfrac{u}{b}\right)=\frac{\Upsilon\left(\frac{u}{b}\right)}{\sqrt{2\pi}\widetilde{\Lambda}^{*}_{\phi\upsilon}\left(\frac{u}{b}\right)} \tag{47}$$

where $\mathring{\Phi}(\frac{u}{b})$ denotes the FT (with its argument scaled by $\frac{1}{b}$) of $\mathring{\phi}(t)$. Equation (47) is determined by the biorthogonality condition

$$\left\langle\mathring{\phi}_{k,M}(t),\phi_{n,M}(t)\right\rangle_{L^2}=\delta_{n-k}. \tag{48}$$

The proof of (47) is given as follows.

Since $\mathring{\phi}_{n,M}(t) \in \mathcal{S}_M(\upsilon)$ for any $n \in \mathbb{Z}$, we may represent $\mathring{\phi}_{0,M}(t)$ as a linear combination of $\{\upsilon_{n,M}(t)\}_{n\in\mathbb{Z}}$, i.e.,

$$\mathring{\phi}(t)e^{-\frac{ja}{2b}t^2} = \sum_{n\in\mathbb{Z}} x[n]\upsilon_{n,M}(t) = x[n]\Theta_{\mathrm{s}}\upsilon(t) \tag{49}$$

with scalar sequence $x[n] \in \ell^2(\mathbb{Z})$. Taking the LCTs of both sides of (49) and applying (20) result in

$$A_b e^{\frac{jd}{2b}u^2}\mathring{\Phi}\left(\frac{u}{b}\right) = \widetilde{X}_M(u)\Upsilon\left(\frac{u}{b}\right) \tag{50}$$

where $\widetilde{X}_M(u)$ denotes the DTLCT of $x[n]$. Next, inserting (49) into (48) leads to

$$\begin{aligned}\delta_n &= \left\langle \mathring{\phi}_{0,M}(t), \phi_{n,M}(t)\right\rangle_{L^2}\\ &= \left\langle \sum_{k\in\mathbb{Z}} r[k]\upsilon_{k,M}(t), \phi_{n,M}(t)\right\rangle_{L^2}\\ &= \sum_{k\in\mathbb{Z}} x[k]\left\langle \upsilon_{k,M}(t), \phi_{n,M}(t)\right\rangle_{L^2}\\ &= \sum_{k\in\mathbb{Z}} x[k]\lambda_{\upsilon\phi}[n-k]e^{-\frac{ja}{2b}(n^2-k^2)}\\ &= x[n]\Theta_{\mathrm{f}}\lambda_{\upsilon\phi}[n].\end{aligned} \tag{51}$$

Taking the DTLCTs of both sides of (51) and using (18) yield

$$A_b e^{\frac{jd}{2b}u^2} = \sqrt{2\pi}\widetilde{X}_M(u)\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right). \tag{52}$$

By combining (52) and (50), (47) can be established.

2) Connection with Reproducing Kernel Hilbert Spaces: We first briefly introduce the concept of reproducing kernel Hilbert spaces (RKHS's). A closed vector space $\mathcal{V}$ is a RKHS with reproducing kernel $\kappa(t,t')$ if and only if [37]

$$\forall t_0 \in \mathbb{R}, \quad \kappa(\cdot, t_0) \in \mathcal{V} \tag{53}$$

and

$$\forall f(t) \in \mathcal{V}, \quad \langle f(\cdot), \kappa(t,\cdot)\rangle_{L^2} = f(t). \tag{54}$$

Then, by Theorem 1, we will show that the approximation space $\mathcal{V}_M(\phi)$ associated with the LCT forms a RKHS. We also need $\upsilon(t) = O\left((1+|t|)^{-\epsilon}\right)$ for some $\epsilon > 1/2$. It is easy to see that (46) is equivalent to

$$\begin{aligned}\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M} f(t) &= \left\langle f(\cdot), \sum_{n\in\mathbb{Z}}\mathring{\phi}_{n,M}(\cdot)\phi^*_{n,M}(t)\right\rangle_{L^2}\\ &= \left\langle f(\cdot), \kappa_M(t,\cdot)\right\rangle_{L^2}\end{aligned} \tag{55}$$

where

$$\kappa_M(t,t') = \sum_{k\in\mathbb{Z}}\mathring{\phi}_{k,M}(t')\phi^*_{k,M}(t). \tag{56}$$

By Theorem 1, it is clear that

$$\forall f(t) \in \mathcal{V}_M(\phi), \quad \langle f(\cdot), \kappa_M(t,\cdot)\rangle_{L^2} = f(t) \tag{57}$$

which is the same as (54), except that the kernel $\kappa_M(t,t')$ may be different from $\kappa(t,t')$. Next, we give the derivation of the requirement $\forall t_0 \in \mathbb{R}, \ \ \kappa_M(\cdot, t_0) \in \mathcal{V}_M(\phi)$. Given $t_0 \in \mathbb{R}$, combining (49) and Parseval's identity of the DTFT [2], we can derive

$$\begin{aligned}&\left(\sum_{n\in\mathbb{Z}}\left|\mathring{\phi}_{n,M}(t_0)\right|^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{n\in\mathbb{Z}}\left|\mathring{\phi}(t_0-n)\right|^2\right)^{\frac{1}{2}}\\ &= \left\|\sum_{n\in\mathbb{Z}}\mathring{\phi}(t_0-n)e^{j\frac{u}{b}n}\right\|_{\ell^2}\\ &= \left\|\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} x[l]\upsilon(t_0-n-l)e^{\frac{ja}{2b}l^2}e^{j\frac{u}{b}n}\right\|_{\ell^2}\\ &= \left\|\sum_{l\in\mathbb{Z}} x[l]e^{\frac{ja}{2b}l^2-j\frac{u}{b}l}\sum_{n\in\mathbb{Z}}\upsilon(t_0-n-l)e^{j\frac{u}{b}(n+l)}\right\|_{\ell^2}.\end{aligned} \tag{58}$$

Then, using (13), we have

$$\sum_{l\in\mathbb{Z}} x[l]e^{\frac{ja}{2b}l^2-j\frac{u}{b}l} = A_b^{-1}\widetilde{X}_M(u)e^{-\frac{jd}{2b}u^2}. \tag{59}$$

Substituting (52) in (59) leads to

$$\sum_{l\in\mathbb{Z}} x[l]e^{\frac{ja}{2b}l^2-j\frac{u}{b}l} = \frac{1}{\sqrt{2\pi}\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)}. \tag{60}$$

Since $\epsilon_0 \le |\widetilde{\Lambda}_{\phi\upsilon}\left(\frac{u}{b}\right)| \le \epsilon_1$ in Theorem 1, combining (60) and (58) results in

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}}\left|\mathring{\phi}_{n,M}(t_0)\right|^2\right)^{\frac{1}{2}} &= \left\|\frac{\sum_{n'\in\mathbb{Z}}\upsilon(t_0-n')e^{j\frac{u}{b}n'}}{\sqrt{2\pi}\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)}\right\|_{\ell^2}\\ &\le \frac{\left\|\sum_{n'\in\mathbb{Z}}\upsilon(t_0-n')e^{j\frac{u}{b}n'}\right\|_{\ell^2}}{\|\sqrt{2\pi}\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)\|_0}\\ &= \frac{\left\|\sum_{n'\in\mathbb{Z}}\upsilon(t_0-n')e^{j\frac{u}{b}n'}\right\|_{\ell^2}}{\|\sqrt{2\pi}\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)\|_0}\\ &= \frac{\left(\sum_{n'\in\mathbb{Z}}|\upsilon(t_0-n')|^2\right)^{\frac{1}{2}}}{\|\sqrt{2\pi}\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)\|_0}\end{aligned} \tag{61}$$

where the index transform $n+l = n'$ is used. Clearly, $\left\{\mathring{\phi}_{n,M}(t_0)\right\}_{n\in Z}$ lies in $\ell^2(\mathbb{Z})$ due to the fact that $\upsilon(t) = O\left((1+|t|)^{-\epsilon}\right)$ for some $\epsilon > 1/2$. Then, from (56), we conclude that $\kappa_M(\cdot,t_0) \in \mathcal{V}_M(\phi)$. In particular, if we choose $\phi(t) = \mathrm{sinc}(t)$, the constructed RKHS will reduce to the RKHS for signals bandlimited in the LCT domain, which was derived by Zhao *et. al* in [25].

3) Minimum Error Solution: It is clear that $\tilde{f}(t) = \mathcal{P}_{\mathcal{S}_M}f(t)$ is the orthogonal projection of $f(t)$ onto $\mathcal{S}_M(\upsilon)$ if $\mathcal{V}_M(\phi) = \mathcal{S}_M(\upsilon)$. In this special case, Theorem 1 corresponds to the least squares sampling theorem of the LCT derived in [32]. Therefore, by (24), Theorem 1 provides the minimum error approximation $f(t) \in L^2(\mathbb{R})$ if and only if $\upsilon(t)e^{-\frac{ja}{2b}t^2} \in \mathcal{V}_M(\phi)$. Correspondingly, the equivalent biorthogonal function in (46) is precisely the dual of the basis function $\phi_{n,M}(t) \in \mathcal{V}_M(\phi)$ described in the fourth row of Table I, and the sampling procedure is optimal. Otherwise, we will obtain an approximate solution, and the performance of which depends on the "similarity" of the approximation space $\mathcal{V}_M(\phi)$ and the measurement space $\mathcal{S}_M(\upsilon)$. A detailed discussion of this issue is presented in Section IV.

4) Signal Approximation in Measurement Space: Let $\mathcal{P}_{\mathcal{S}_M} f(t)$ be the orthogonal projection of $f(t)$ on $\mathcal{S}_M(\upsilon)$. Since $\mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t)$ is the projection of $f(t)$ on $\mathcal{V}_M(\phi)$ perpendicular to $\mathcal{S}_M(\upsilon)$, for any $f(t) \in L^2(\mathbb{R})$, it follows that $\mathcal{P}_{\mathcal{S}_M} f(t) = \mathcal{P}_{\mathcal{S}_M} \mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t)$, as shown in Fig. 3. This

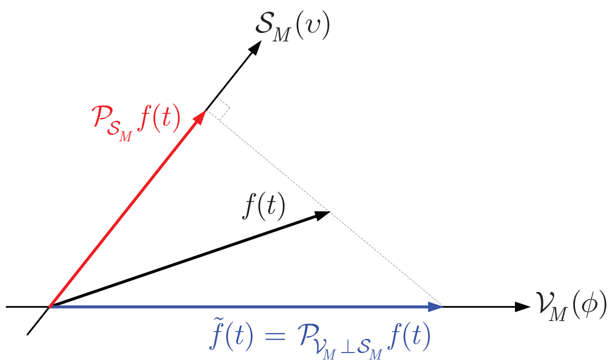

Fig. 3. Graphical representation of function approximations.

implies that the approximation of $\tilde{f}(t)$ in (29) contains all the information necessary to compute $\mathcal{P}_{\mathcal{S}_M} f(t)$. Now, we discuss the computation of $\mathcal{P}_{\mathcal{S}_M} f(t)$ from $\mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t)$ in detail. By using the general equivalent generating function in (24) and its property described in the third row of Table I, the orthogonal generating function $\upsilon_{\text{ortho}}(t)$ of $\mathcal{S}_M(\upsilon)$ satisfies

$$\upsilon_{\text{ortho}}(t) e^{-\frac{ja}{2b}t^2} = \sum_{n \in \mathbb{Z}} q_1[n] \upsilon_{n,M}(t) \tag{62}$$

where expansion coefficients $q_1[n]$ is the inverse DTLCT of $\left(\sqrt{2\pi}\widetilde{\Lambda}_{\upsilon\upsilon}\left(\frac{u}{b}\right)\right)^{-\frac{1}{2}} A_b e^{\frac{jd}{2b}u^2}$. For simplicity, let

$$\tilde{q}[n] \triangleq q[n] e^{\frac{ja}{2b}n^2} \tag{63}$$

denote the chirp-modulated version of a sequence $q[n]$. Further, from (62), the orthogonal basis function of $\mathcal{S}_M(\upsilon)$ is given by

$$\begin{aligned} \upsilon_{\text{ortho},n,M}(t) &= \upsilon_{\text{ortho}}(t-n) e^{-\frac{ja}{2b}(t^2-n^2)} \\ &= \sum_{k \in \mathbb{Z}} \tilde{q}_1[k] \upsilon(t-n-k) e^{-\frac{ja}{2b}(t^2-n^2)}. \end{aligned} \tag{64}$$

The coefficients of orthogonal projection of $\mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t)$ onto $\mathcal{S}_M(\upsilon)$ can then be calculated as

$$\begin{aligned} \varsigma[n] &= \left\langle \mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t), \upsilon_{\text{ortho},n,M}(t) \right\rangle_{L^2} \\ &= \left\langle \sum_{m \in \mathbb{Z}} c_2[m] \phi_{m,M}(t), \upsilon_{\text{ortho},n,M}(t) \right\rangle_{L^2} \\ &= e^{-\frac{ja}{2b}n^2} \left(\tilde{c}_2 \star (\tilde{q}_1 * \lambda_{\phi\upsilon}) \right)[n] \end{aligned} \tag{65}$$

where $\star$ and $*$ denote the ordinary correlation and convolution operators, respectively. Therefore, we have

$$\begin{aligned} \forall f(t) \in L^2(\mathbb{R}),\ \mathcal{P}_{\mathcal{S}_M} f(t) &= \mathcal{P}_{\mathcal{S}_M} \mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t) \\ &= \sum_{n \in Z} \varsigma[n] \upsilon_{\text{ortho},n,M}(t). \end{aligned} \tag{66}$$

5) Generalization of Shannon's Sampling Theory for LCT: Theorem 1 provides a restatement of Shannon's sampling theorem associated with the LCT for real measured signal values, i.e., samples obtained after convolution with the impulse response of the sensor. Also, it can be viewed as a generalization of Shannon's sampling theory for the LCT. This may be interpreted twofold. Equation (36) states that functions included in the approximation space $\mathcal{V}_M(\phi)$ are left unchanged. An equivalent formulation of this property is that any function $f(t) \in \mathcal{V}_M(\phi)$ can be reconstructed without any loss from its sampled measured values. This statement is very similar to Shannon's sampling theorem for the LCT [20]. The main distinction is that we are now talking about measured values instead of signal samples. The class of functions considered here is also more general than the family of LCT-bandlimited signals given in (5). On the other hand, when $\phi(t) = \upsilon(t) = \text{sinc}(t)$, it follows that $h[n] = \delta[n]$. In this case, Theorem 1 reduces to Shannon's sampling theorem for the LCT [20]. In our previous work [32], we studied the relationship between Shannon's sampling theorem and sampling procedures in function spaces associated with the LCT in detail. In addition, if $M = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)$, Theorem 1 reduces to Shannon's sampling theorem for the FRFT [38]. Further, if $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, it is identical to classical Shannon's sampling theorem [29].

C. Input/Output Coherence Measures

An important practical issue is how much better the approximation $\mathcal{P}_{\mathcal{V}_M \perp \mathcal{S}_M} f(t)$ is when compared to the optimal least squares solution $\mathcal{P}_{\mathcal{V}_M} f(t)$. To give a quantitative answer to this question, we will define an input-output coherence function that measures the "similarity" between the approximation space $\mathcal{V}_M(\phi)$ and the measurement space $\mathcal{S}_M(\upsilon)$.

Our initial assumption is that both $\phi(t)$ and $\upsilon(t)$ satisfy the Riesz basis condition (23). Thus, it is possible to construct the orthogonal generating functions $\phi_{\text{ortho}}(t)$ and $\upsilon_{\text{ortho}}(t)$, see the third row of Table I. Then, we can derive a normalized form of the cross-correlation sequence $\lambda_{\phi\upsilon}[k]$ as

$$\gamma_{\phi\upsilon}[k] = \langle \phi_{\text{ortho}}(t), \upsilon_{\text{ortho}}(t-k) \rangle_{L^2}. \tag{67}$$

Following the derivation of $\upsilon_{\text{ortho}}(t)$ in (62), the orthogonal generating function $\phi_{\text{ortho}}(t)$ of $\mathcal{V}_M(\phi)$ is determined by

$$\phi_{\text{ortho}}(t) e^{-\frac{ja}{2b}t^2} = \sum_{n \in \mathbb{Z}} q_2[n] \phi_{n,M}(t) \tag{68}$$

where expansion coefficients $q_2[n]$ is the inverse DTLCT of $\left(\sqrt{2\pi}\widetilde{\Lambda}_{\phi\phi}\left(\frac{u}{b}\right)\right)^{-\frac{1}{2}} A_b e^{\frac{jd}{2b}u^2}$. It follows from (68) and (63) that

$$\phi_{\text{ortho}}(t) = \sum_{n \in \mathbb{Z}} \tilde{q}_2[n] \phi(t-n). \tag{69}$$

Combining (62), (67), and (69), we have

$$\gamma_{\phi\upsilon}[k] = \left(\tilde{q}_2 \star (\tilde{q}_1^* * \lambda_{\phi\upsilon}) \right)[k]. \tag{70}$$

Taking the DTFTs (with its argument scaled by $\frac{1}{b}$) of both sides of (70) and using (28) yield

$$\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right) = \frac{\widetilde{\Lambda}^*_{\phi\upsilon}\left(\frac{u}{b}\right)}{\sqrt{2\pi \widetilde{\Lambda}_{\phi\phi}\left(\frac{u}{b}\right) \widetilde{\Lambda}_{\upsilon\upsilon}\left(\frac{u}{b}\right)}} \tag{71}$$

where $\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)$ denotes the DTFT (with its argument scaled by $\frac{1}{b}$) of $\gamma_{\phi\upsilon}[k]$. To get a measure that is symmetrical with respect to $\phi(t)$ and $\upsilon(t)$, we take the modulus of $\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)$. This allows us to define the spectral coherence function

$$\begin{aligned}\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\tfrac{u}{b}\right)\right| = &\frac{\left|\sum_{k\in\mathbb{Z}}\Phi(\frac{u}{b}+2k\pi)\Upsilon^*(\frac{u}{b}+2k\pi)\right|}{\sqrt{\sum_{k\in\mathbb{Z}}|\Phi(\frac{u}{b}+2k\pi)|^2}}\\ &\times\frac{1}{\sqrt{2\pi}\sqrt{\sum_{k\in\mathbb{Z}}|\Upsilon(\frac{u}{b}+2k\pi)|^2}}\end{aligned} \tag{72}$$

which can be viewed as a function of the FTs (with its argument scaled by $\frac{1}{b}$) of the generating functions $\phi(t)$ and $\upsilon(t)$ using Poisson's summation formula. This spectral coherence function also takes the form of a correlation coefficient between the spectral components of $\phi_{\text{ortho}}(t)$ and $\upsilon_{\text{ortho}}(t)$. Some relevant properties of it are described as follows:

Theorem 2: If the generating functions $\phi(t)$ and $\upsilon(t)$ satisfy the Riesz basis condition (23), then the spectral coherence function $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ has the following properties:

(i) $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ is periodic and independent of a particular choice of the generating functions for $\mathcal{V}_M(\phi)$ and $\mathcal{S}_M(\upsilon)$.

(ii) $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right| \leq \frac{1}{\sqrt{2\pi}}$, $\forall u\in\mathbb{R}$. The equality is achieved if and only if $\mathcal{V}_M(\phi)=\mathcal{S}_M(\phi)$.

(iii) If the generating functions $\phi(t)$ and $\upsilon(t)$ lie in $L^1(\mathbb{R})$, and their scaled FTs $\Phi(\frac{u}{b})$ and $\Upsilon(\frac{u}{b})$ both decay like $O(|u|^{-\varepsilon})$, $\varepsilon>\frac{1}{2}$, then $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ is continuous.

Proof: (i) From (72), it is easy to see that

$$\begin{aligned}\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\tfrac{u+2\pi b}{b}\right)\right| = &\frac{\left|\sum_{k'\in\mathbb{Z}}\Phi(\frac{u}{b}+2k'\pi)\Upsilon^*(\frac{u}{b}+2k'\pi)\right|}{\sqrt{\sum_{k'\in\mathbb{Z}}|\Phi(\frac{u}{b}+2k'\pi)|^2}}\\ &\times\frac{1}{\sqrt{2\pi}\sqrt{\sum_{k'\in\mathbb{Z}}|\Upsilon(\frac{u}{b}+2k'\pi)|^2}}\\ =&\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\tfrac{u}{b}\right)\right|\end{aligned} \tag{73}$$

where the index transform $k+1=k'$ is used. Thus, $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ is a periodic function with period $2\pi b$. The invariance property can be verified by computing the spectral coherence for general equivalent generating functions $\phi_{\text{eq}}(t)$ and $\upsilon_{\text{eq}}(t)$ defined by (24). To be specific, using (24) and (20), the FT $\Phi_{\text{eq}}(\frac{u}{b})$ (with its argument scaled by $\frac{1}{b}$) of the general equivalent generating function $\phi_{\text{eq}}(t)$ of $\mathcal{V}_M(\phi)$ can be derived as

$$\Phi_{\text{eq}}\left(\tfrac{u}{b}\right)=A_b^{-1}\widetilde{Q}_M(u)e^{-\frac{jd}{2b}u^2}\Phi\left(\tfrac{u}{b}\right) \tag{74}$$

where $\widetilde{Q}_M(u)$ denotes the DTLCT of $q[n]$ defined in (24). Combining (74) and (16), we can see that for any $k\in\mathbb{Z}$

$$\Phi_{\text{eq}}\left(\tfrac{u}{b}+2k\pi\right)=A_b^{-1}\widetilde{Q}_M(u)e^{-\frac{jd}{2b}u^2}\Phi\left(\tfrac{u}{b}+2k\pi\right). \tag{75}$$

Meanwhile, from (24), there exists a general equivalent generating function $\upsilon_{\text{eq}}(t)$ for $\mathcal{S}_M(\upsilon)$ such that

$$\begin{aligned}\upsilon_{\text{eq}}(t)e^{-\frac{ja}{2b}t^2}&=\sum_{n\in\mathbb{Z}}y[n]\upsilon_{n,M}(t)\\ &=y[n]\Theta_{\text{s}}\upsilon(t)\end{aligned} \tag{76}$$

where expansion coefficients $y[n]$ satisfies the condition in (25). Let $\Upsilon_{\text{eq}}(\frac{u}{b})$ and $\widetilde{Y}_M(u)$ denote the FT (with its argument scaled by $\frac{1}{b}$) of $\upsilon_{\text{eq}}(t)$ and the DTLCT of $y[n]$, respectively. Using (76) and (20), we have

$$\Upsilon_{\text{eq}}\left(\tfrac{u}{b}\right)=A_b^{-1}\widetilde{Y}_M(u)e^{-\frac{jd}{2b}u^2}\Upsilon\left(\tfrac{u}{b}\right) \tag{77}$$

from which together with (16), it follows that

$$\Upsilon_{\text{eq}}\left(\tfrac{u}{b}+2k\pi\right)=A_b^{-1}\widetilde{Y}_M(u)e^{-\frac{jd}{2b}u^2}\Upsilon\left(\tfrac{u}{b}+2k\pi\right). \tag{78}$$

Then, from (72), the spectral coherence function for general equivalent generating functions $\phi_{\text{eq}}(t)$ and $\upsilon_{\text{eq}}(t)$ is given by

$$\begin{aligned}\left|\widetilde{\Gamma}_{\phi_{\text{eq}}\upsilon_{\text{eq}}}\left(\tfrac{u}{b}\right)\right| = &\frac{\left|\sum_{k\in\mathbb{Z}}\Phi_{\text{eq}}(\frac{u}{b}+2k\pi)\Upsilon_{\text{eq}}^*(\frac{u}{b}+2k\pi)\right|}{\sqrt{\sum_{k\in\mathbb{Z}}|\Phi_{\text{eq}}(\frac{u}{b}+2k\pi)|^2}}\\ &\times\frac{1}{\sqrt{2\pi}\sqrt{\sum_{k\in\mathbb{Z}}|\Upsilon_{\text{eq}}(\frac{u}{b}+2k\pi)|^2}}.\end{aligned} \tag{79}$$

Inserting (75) and (78) into (79) results in

$$\left|\widetilde{\Gamma}_{\phi_{\text{eq}}\upsilon_{\text{eq}}}\left(\tfrac{u}{b}\right)\right|=\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\tfrac{u}{b}\right)\right| \tag{80}$$

which states that $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ is independent of a particular choice of generating functions for spaces $\mathcal{V}_M(\phi)$ and $\mathcal{S}_M(\upsilon)$.

(ii): For a fixed value of u, using Schwarz's inequality yields

$$\begin{aligned}&\left|\sum_{k\in\mathbb{Z}}\Phi(\tfrac{u}{b}+2k\pi)\Upsilon^*(\tfrac{u}{b}+2k\pi)\right|^2\\ &\leq\left(\sum_{k\in\mathbb{Z}}\left|\Phi(\tfrac{u}{b}+2k\pi)\right|^2\right)\left(\sum_{k\in\mathbb{Z}}\left|\Upsilon(\tfrac{u}{b}+2k\pi)\right|^2\right)\end{aligned} \tag{81}$$

which together with (72) implies that $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|\leq\frac{1}{\sqrt{2\pi}}$. The equality holds for some LCT frequency u_0 if and only if there exists a constant $\widetilde{P}$ (which depends on u_0) such that

$$\Phi(\tfrac{u_0}{b}+2k\pi)=\widetilde{P}(\tfrac{u_0}{b})\Upsilon(\tfrac{u_0}{b}+2k\pi). \tag{82}$$

Then, the condition

$$\left|\Gamma_{\phi\upsilon}\left(\tfrac{u}{b}\right)\right|=\frac{1}{\sqrt{2\pi}},\ \forall u\in\mathbb{I} \tag{83}$$

is equivalent to

$$\forall k\in\mathbb{Z},\quad \Phi(\tfrac{u}{b}+2k\pi)=\widetilde{P}(\tfrac{u}{b})\Upsilon(\tfrac{u}{b}+2k\pi) \tag{84}$$

which implies that

$$\forall k\in\mathbb{Z},\quad \widetilde{P}(\tfrac{u}{b})=\widetilde{P}(\tfrac{u}{b}+2k\pi). \tag{85}$$

Next, combining (84) and (85) results in

$$\sum_{k\in\mathbb{Z}}\left|\Phi(\tfrac{u}{b}+2k\pi)\right|^2=\left|\widetilde{P}(\tfrac{u}{b})\right|^2\sum_{k\in\mathbb{Z}}\left|\Upsilon(\tfrac{u}{b}+2k\pi)\right|^2. \tag{86}$$

Since $\phi(t)$ and $\upsilon(t)$ both satisfy the Riesz basis condition (23) with bounds (A_1,B_1) and (A_2,B_2) for $\mathcal{V}_M(\phi)$ and $\mathcal{S}_M(\upsilon)$, respectively, we have

$$0<\frac{A_1}{B_2}\leq\left|\widetilde{P}(\tfrac{u}{b})\right|^2\leq\frac{B_1}{A_2}. \tag{87}$$

Based on the above facts, we can conclude that $\widetilde{P}(\frac{u}{b})\in L^2(\mathbb{I})$. This means that we can take the inverse DTFT (with its argument scaled by $\frac{1}{b}$) of $\widetilde{P}(\frac{u}{b})$ in $L^2(\mathbb{I})$ denoted by $p[n]$. Then, from (84), we derive

$$\phi(t)=\sum_{n\in\mathbb{Z}} p[n]\upsilon(t-n) \tag{88}$$

which leads to

$$\phi(t)e^{-\frac{ja}{2b}t^2}=\sum_{n\in\mathbb{Z}} \tilde{p}^*[n]\upsilon_{n,M}(t). \tag{89}$$

Then, it follows from (24) that $\phi(t)$ and $\upsilon(t)$ generate the same function space, i.e., $\mathcal{V}_M(\phi)=\mathcal{S}_M(\upsilon)$.

(iii): By the Riemann-Lebesgue Lemma, functions $\Phi(\frac{u}{b})$ and $\Upsilon(\frac{u}{b})$ are both continuous. If they also decay like $O(|u|^{-\varepsilon})$, $\varepsilon>\frac{1}{2}$, then the series

$$\begin{aligned}\sum_{k\in\mathbb{Z}} &\Phi(\tfrac{u}{b}+2k\pi)\Upsilon^*(\tfrac{u}{b}+2k\pi)\\ &\le \sum_{k\in\mathbb{Z}}\left|\Phi(\tfrac{u}{b}+2k\pi)\Upsilon^*(\tfrac{u}{b}+2k\pi)\right|\\ &\le \text{const}\sum_{k=1}^{+\infty}|k|^{-2\varepsilon}\end{aligned} \tag{90}$$

are absolutely convergent independently of u. Therefore, these series are continuous on $\mathbb{I}$. Since the denominators on the right hand side of (72) are bounded and nonvanishing, the spectral coherence function $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ is continuous as well. This completes the proof of Theorem 2. □

The spectral coherence function $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ can be used to compute an average measure of coherence between $\mathcal{V}_M(\phi)$ and $\mathcal{S}_M(\upsilon)$. Using Parseval's identity of the DTFT [2], we arrive at

$$0\le\left(\int_{\mathbb{I}}\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\tfrac{u}{b}\right)\right|^2 d\left(\tfrac{u}{b}\right)\right)^{\frac{1}{2}}=\|\gamma_{\phi\upsilon}[k]\|_{\ell^2}\le 1. \tag{91}$$

In particular, note in (ii) of Theorem 2, $\left|\widetilde{\Gamma}_{\phi\upsilon}\left(\frac{u}{b}\right)\right|$ has the property that it is equal to $\frac{1}{\sqrt{2\pi}}$ if and only if $\mathcal{V}_M(\phi)=\mathcal{S}_M(\phi)$. In this case, signal approximations $\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)$ and $\mathcal{P}_{\mathcal{V}_M}f(t)$ are equivalent.

IV. Error Estimates

If the generating function $\upsilon(t)$ of $\mathcal{S}_M(\upsilon)$ satisfies the condition $\upsilon(t)e^{-\frac{ja}{2b}t^2}\in\mathcal{V}_M(\phi)$, then we can derive the minimum error approximation based upon Theorem 1. Otherwise, Theorem 1 will provide us with an approximate solution. Now, we give a theoretical performance bound that corresponds to a worst case scenario via the concept of the minimal and maximal angles between two subspaces.

To derive the maximal angle between the measurement space $\mathcal{S}_M(\upsilon)$ and the approximation space $\mathcal{V}_M(\phi)$, we use the orthogonal representation of a function $g(t)\in\mathcal{S}_M(\upsilon)$ and the orthogonal projection $\mathcal{P}_{\mathcal{V}_M}g(t)$ of $g(t)$ onto $\mathcal{V}_M(\upsilon)$, i.e.,

$$g(t)=\sum_{l\in\mathbb{Z}}\omega[l]\upsilon_{\text{ortho},l,M}(t) \tag{92}$$

$$\mathcal{P}_{\mathcal{V}_M}g(t)=\sum_{k\in\mathbb{Z}}\rho[k]\phi_{\text{ortho},k,M}(t) \tag{93}$$

where $\{\omega[l]\}_{l\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ and $\{\rho[k]\}_{k\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$. The most important property of the orthogonal representation for our purpose is that the L^2-norm of a function is also equal to the ℓ^2-norm of its coefficients in the orthogonal representation. Then, combining the linearity of the orthogonal projection operator $\mathcal{P}_{\mathcal{V}_M}$, (92), and (67) gives rise to

$$\begin{aligned}\mathcal{P}_{\mathcal{V}_M}g(t)&=\sum_{l\in\mathbb{Z}}\omega[l]\mathcal{P}_{\mathcal{V}_M}\upsilon_{\text{ortho},l,M}(t)\\ &=\sum_{l\in\mathbb{Z}}\omega[l]\sum_{k\in\mathbb{Z}}\langle\upsilon_{\text{ortho},l,M}(t),\phi_{\text{ortho},k,M}(t)\rangle\\ &\quad\times\phi_{\text{ortho},k,M}(t)\\ &=\sum_{l\in\mathbb{Z}}\omega[l]\sum_{k\in\mathbb{Z}}\langle\upsilon_{\text{ortho}}(t-l),\phi_{\text{ortho}}(t-k)\rangle\\ &\quad\times e^{-\frac{ja}{2b}(k^2-l^2)}\phi_{\text{ortho},k,M}(t)\\ &=\sum_{k\in\mathbb{Z}}\left(\omega\Theta_{\text{f}}\gamma_{\upsilon\phi}\right)[k]\phi_{\text{ortho},k,M}(t)\end{aligned} \tag{94}$$

which clearly shows that the coefficients of $\mathcal{P}_{\mathcal{V}_M}g(t)$ in (93) can be derived as

$$\rho[k]=\left(\omega\Theta_{\text{f}}\gamma_{\upsilon\phi}\right)[k] \tag{95}$$

from which together with (18) it follows that

$$\widetilde{\mathcal{L}}^M\{\rho[k]\}(u)=\sqrt{2\pi}\widetilde{\Omega}_M(u)\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b}) \tag{96}$$

where $\widetilde{\Omega}_M(u)$ denotes the DTLCT of $\omega[k]$. Next, from (15), it follows that

$$\begin{aligned}\|\mathcal{P}_{\mathcal{V}_M}g\|_{L^2}^2&=\|\rho\|_{\ell^2}^2\\ &=\int_{\mathbb{I}}\left|\sqrt{2\pi}\widetilde{\Omega}_M(u)\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b})\right|^2du\\ &=\int_{\mathbb{I}}\left|\widetilde{\Omega}_M(u)\right|^2\cdot\left|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b})\right|^2du\\ &\ge\left\|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b})\right\|_0^2\cdot\int_{\mathbb{I}}\left|\widetilde{\Omega}_M(u)\right|^2du\\ &=\left\|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b})\right\|_0^2\cdot\|g\|_{L^2}^2\end{aligned} \tag{97}$$

which together with (10) implies that

$$\begin{aligned}\inf_{g\in\mathcal{S}_M\backslash\{0\}}\frac{\|\mathcal{P}_{\mathcal{V}_M}g\|_{L^2}}{\|g\|_{L^2}}&=\left\|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\tfrac{u}{b})\right\|_0\\ &=R(\mathcal{S}_M,\mathcal{V}_M).\end{aligned} \tag{98}$$

Note that if condition (iii) of Theorem 2 holds, i.e., $\left|\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})\right|$ is continuous on $\mathbb{I}$, then the essential infimum $\left\|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})\right\|_0$ will reduce to the minimum of $\left|\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})\right|$. Since $\left|\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})\right|$ is symmetrical, we can also restrict the analysis to the interval $[0,+\pi b]$. Note also that the bound given by (98) is symmetrical with respect to the generating functions $\phi(t)$ and $\upsilon(t)$. Accordingly, it follows that $R(\mathcal{S}_M,\mathcal{V}_M)=R(\mathcal{V}_M,\mathcal{S}_M)$.

Based on the above results, we now address the crucial issue of how close our approximation $\tilde{f}(t)=\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)$ is to the optimal estimate $\mathcal{P}_{\mathcal{V}_M}f(t)$, which is the orthogonal projection of $f(t)$ onto $\mathcal{V}_M(\phi)$. The question is of importance because the minimum error solution is in general not accessible. However, we would like to have some guarantee that the approximation produced by Theorem 1 is reasonably close to the best possible estimate.

Theorem 3: Let $\mathcal{S}_M(\upsilon)$ and $\mathcal{V}_M(\phi)$ be two closed subspaces of $L^2(\mathbb{R})$ with bases $\{\upsilon_{n,M}(t)\}_{n\in\mathbb{Z}}$ and $\{\phi_{n,M}(t)\}_{n\in\mathbb{Z}}$, respectively. Let $\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}$ be the oblique projection operator onto $\mathcal{V}_M(\phi)$ along $\mathcal{S}_M^{\perp}(\upsilon)$, and $\mathcal{P}_{\mathcal{V}_M}$ be the orthogonal projection operator onto $\mathcal{V}_M(\phi)$. Then, for any function $f(t)\in L^2(\mathbb{R})$, it follows that

$$1 \le \frac{\|f-\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f\|_{L^2}}{\|f-\mathcal{P}_{\mathcal{V}_M}f\|_{L^2}} \le \frac{1}{\|\sqrt{2\pi}\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})\|_0} \tag{99}$$

where $\widetilde{\Gamma}_{\upsilon\phi}(\frac{u}{b})$ is defined in (72).

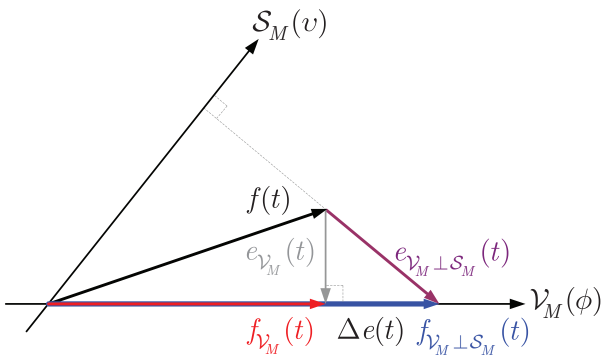

Fig. 4. Graphical representation of the various approximations and errors in the function spaces.

Proof: Since $\mathcal{P}_{\mathcal{V}_M}$ is the orthogonal projection operator onto $\mathcal{V}_M(\phi)$, it follows that $\|f-\mathcal{P}_{\mathcal{V}_M}f\|_{L^2} \le \|f-\varphi\|_{L^2}$, $\forall\varphi(t)\in\mathcal{V}_M(\phi)$. This clearly shows that the left hand side inequality of (99) holds. Next, to derive the right hand side inequality, we define the following errors

$$\begin{cases} e_{\mathcal{V}_M\perp\mathcal{S}_M}(t) = f(t) - f_{\mathcal{V}_M\perp\mathcal{S}_M}(t) \\ e_{\mathcal{V}_M}(t) = f(t) - f_{\mathcal{V}_M}(t) \\ \Delta e(t) = f_{\mathcal{V}_M\perp\mathcal{S}_M}(t) - f_{\mathcal{V}_M}(t) \end{cases} \tag{100}$$

where $f_{\mathcal{V}_M\perp\mathcal{S}_M}(t) = \mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)$ and $f_{\mathcal{V}_M}(t) = \mathcal{P}_{\mathcal{V}_M}f(t)$. A graphical illustration of these errors is shown in Fig. 4. It is easy to see that $e_{\mathcal{V}_M\perp\mathcal{S}_M}(t)\in\mathcal{S}_M^{\perp}(\upsilon)$, $e_{\mathcal{V}_M}(t)\in\mathcal{V}_M^{\perp}(\phi)$, and $\Delta e(t)\in\mathcal{V}_M(\phi)$. Clearly, $e_{\mathcal{V}_M\perp\mathcal{S}_M}(t) = \Delta e(t) + e_{\mathcal{V}_M}(t)$, in addition $\Delta e(t)\perp e_{\mathcal{V}_M}(t)$. Then, from the Projection Theorem [39], we obtain

$$\Delta e(t) = \mathcal{P}_{\mathcal{V}_M} e_{\mathcal{V}_M\perp\mathcal{S}_M}(t) \tag{101}$$

$$e_{\mathcal{V}_M}(t) = e_{\mathcal{V}_M\perp\mathcal{S}_M}(t) - \mathcal{P}_{\mathcal{V}_M} e_{\mathcal{V}_M\perp\mathcal{S}_M}(t). \tag{102}$$

This result shows that $e_{\mathcal{V}_M}(t)$ can be viewed as the orthogonal projection of $e_{\mathcal{V}_M\perp\mathcal{S}_M}(t)\in\mathcal{S}_M^{\perp}(\upsilon)$ on $\mathcal{V}_M^{\perp}(\phi)$. Therefore, using (10), we can derive

$$\begin{aligned} &\inf_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|e_{\mathcal{V}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} \\ &= \inf_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|(\mathcal{I}-\mathcal{P}_{\mathcal{V}_M})e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} \\ &= \inf_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|\mathcal{P}_{\mathcal{V}_M^{\perp}}e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} \\ &= R(\mathcal{S}_M^{\perp},\mathcal{V}_M^{\perp}) \end{aligned} \tag{103}$$

where $\mathcal{I}$ denotes the identity operator. Combining (103) and (11), we can see that

$$\inf_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|e_{\mathcal{V}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} = R(\mathcal{S}_M,\mathcal{V}_M) \tag{104}$$

from which along with (98), the right hand side inequality of (99) is derived. This completes the proof of Theorem 3. □

Note that the right hand side of (99) corresponds to the worst possible case, and this bound may not necessarily reflect what really happens in practice. A more realistic estimate of the error ratio may be $\|\gamma_{\phi\upsilon}[k]\|_{\ell^2}^{-1}$, where $r_{\phi\upsilon}[k]$ is the global coherence measure defined by (91). This latter measure is an average performance index in the sense that it weights to all LCT-frequency components equally. It also corresponds to the case of an impulse or white noise signal. Further, we have the property that $\|\gamma_{\phi\upsilon}[k]\|_{\ell^2}=1$ if and only if $R(\mathcal{S}_M,\mathcal{V}_M)=1$.

Based upon Theorem 3, a measure of the maximum discrepancy between $\mathcal{P}_{\mathcal{V}_M}f(t)$ and $\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)$ is given by

$$\begin{aligned} \sup_{f-\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f\in\mathcal{S}_M^{\perp}\setminus\{0\}} &\frac{\|\mathcal{P}_{\mathcal{V}_M}f-\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f\|_{L^2}}{\|f-\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f\|_{L^2}} \\ &= \sqrt{1-R^2(\mathcal{S}_M,\mathcal{V}_M)}. \end{aligned} \tag{105}$$

Particularly, note that if $\|r_{\phi\upsilon}[k]\|_{\ell^2}=1$, i.e., $R(\mathcal{S}_M,\mathcal{V}_M)=1$, then $\mathcal{P}_{\mathcal{V}_M}f(t)-\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)=0$. This means that the corresponding approximations $\mathcal{P}_{\mathcal{V}_M\perp\mathcal{S}_M}f(t)$ and $\mathcal{P}_{\mathcal{V}_M}f(t)$ are equivalent. The proof of (105) is as follows.

From (100) and (101), (105) can be rewritten as

$$\begin{aligned} &\sup_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|\Delta e\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} \\ &= \sup_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|\mathcal{P}_{\mathcal{V}_M}e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} \end{aligned} \tag{106}$$

from which along with (9) it follows that

$$\begin{aligned} \sup_{e_{\mathcal{V}_M\perp\mathcal{S}_M}\in\mathcal{S}_M^{\perp}\setminus\{0\}} \frac{\|\mathcal{P}_{\mathcal{V}_M}e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}}{\|e_{\mathcal{V}_M\perp\mathcal{S}_M}\|_{L^2}} &= S(\mathcal{S}_M^{\perp},\mathcal{V}_M) \\ &= S(\mathcal{V}_M,\mathcal{S}_M^{\perp}). \end{aligned} \tag{107}$$

Then, combining (107) and (12) leads to (105).

V. Numerical Results and Applications

In this section, we want to see a numerical simulation and several potential applications of the derived results.

We first give a numerical example. Suppose that $f(t)$ is known to be a B-spline of degree 2 with knots at the integers. Such a signal can be expressed as

$$f(t) = \sum_{n\in\mathbb{Z}} d[n]\beta^2(t-n) \tag{108}$$

where $\{d[n]\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$, and $\beta^2(t)$ is the B-spline function of degree 2. It is clear that $f(t)$ belongs to the approximation space $\mathcal{V}_M(\phi)$ with its generating function $\phi(t)=\beta^2(t)$ and parameter $M=\left(\begin{smallmatrix}0 & 1\\ -1 & 2\end{smallmatrix}\right)$. We wish to recover $f(t)$ from average of its values over the intervals $[n-\frac{1}{2}, n+\frac{1}{2}]$, $n\in\mathbb{Z}$. This setting corresponds to sampling at times $t=n$ at the output of the prefilter $\upsilon(t)=\beta^0(t)$, where $\beta^0(t)$ is the B-spline function of degree 0.

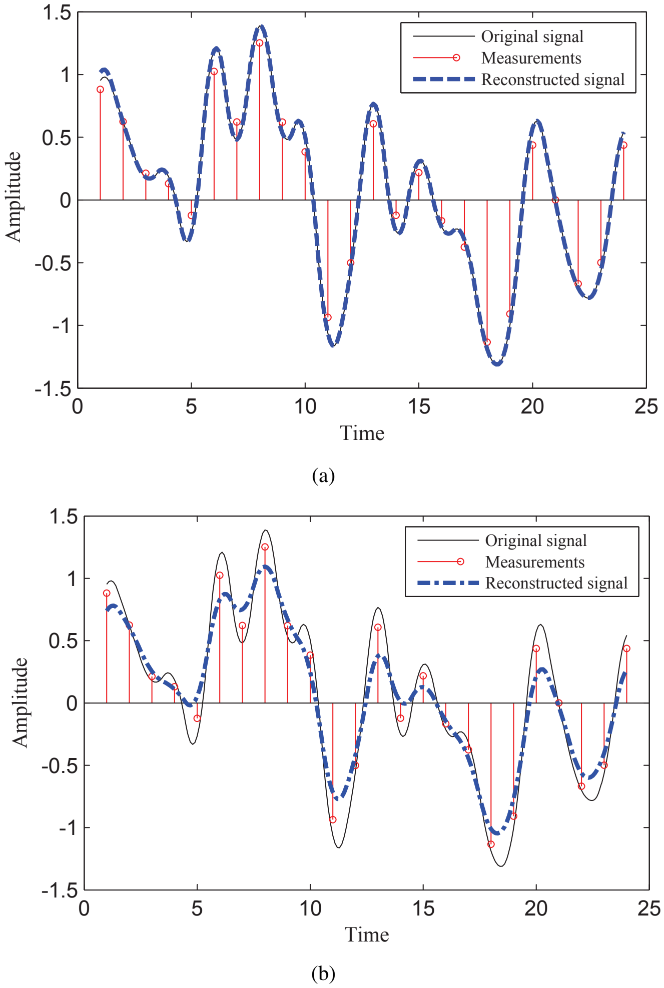

Fig. 5. Reconstruction results: (a) using the proposed sampling scheme; (b) using the sampling scheme of [32] shown in Fig. 1(c).

Using the proposed sampling scheme of Fig. 2, perfect recovery of $f(t)$ can be achieved by passing the measurements $\{c_1[n]\}_{n\in\mathbb{Z}}$ through the digital correction filter $h[n]$ which is the inverse DTFT (with its argument scaled by $\frac{1}{b}$) of

$$\widetilde{H}\left(\tfrac{u}{b}\right) = \frac{1}{2\pi\widetilde{\Lambda}_{\beta^2\beta^0}\left(\frac{u}{b}\right)} = \frac{4}{\sqrt{2\pi}\left(\cos(\frac{u}{b})+3\right)}. \tag{109}$$

The reconstruction results obtained by using the proposed sampling scheme of Fig. 2 with (109) and the scheme introduced in [32] as illustrated in Fig. 1(c) are shown in Fig. 5(a) and Fig. 5(b), respectively. The normalized mean-square error (NMSE) of the proposed sampling scheme is 1.35×10^{-3}, where the NMSE is defined as $\mathrm{NMSE} = \frac{\|\tilde{f}(t)-f(t)\|^2}{\|f(t)\|^2}$, and $\tilde{f}(t)$ denotes the reconstructed signal. By comparison, when using the sampling scheme of [32], the NMSE is 0.2934. Evidently, the digital filtering stage is required in order to perfectly restore the signal from its filtered samples.

Next, we discuss applications of the derived results. The derived theory provides an alternative formulation for the basic deconvolution problem. Towards this end, we consider a measurement device with an impulse response $\upsilon(t)$ and assume that the signal reconstruction is bandlimited with $\phi(t) = \mathrm{sinc}(t)$. From Theorem 1, the DTFT of the correction filter $h[n]$ is given by

$$\widetilde{H}\left(\tfrac{u}{b}\right) = \frac{1}{\Upsilon(\frac{u}{b})}, \quad u\in\mathbb{I} \tag{110}$$

which is identical to the traditional inverse filter solution in the LCT domain [2]. Evidently, this operator is stable by supposing that the LCT-frequency response $\Upsilon(\frac{u}{b})$ of the measurement device satisfies $\Upsilon(\frac{u}{b})\neq 0$ for all $u\in\mathbb{I}$. If $\upsilon(t)$ satisfies the Riesz basis condition (23), the spectral coherence function can be derived as

$$\left|\widetilde{\Gamma}_{\mathrm{sinc}\,\upsilon}\left(\tfrac{u}{b}\right)\right| = \frac{\left|\Upsilon(\frac{u}{b})\right|}{\sqrt{2\pi}\sqrt{\sum_{k\in\mathbb{Z}}|\Upsilon(\frac{u}{b}+2k\pi)|^2}} \tag{111}$$

for $u\in\mathbb{I}$. It is easy to see that (111) is identical to that of the LCT-frequency response of the orthogonal generating function $\upsilon_{\text{ortho}}(t)$ defined by (62), i.e., $|\widetilde{\Gamma}_{\mathrm{sinc}\,\upsilon}\left(\frac{u}{b}\right)| = |\Upsilon_{\text{ortho}}(\frac{u}{b})|$. If $\upsilon(t)$ is bandlimited in the LCT domain, i.e., $\upsilon(t)e^{-\frac{ja}{2b}t^2}\in\mathcal{V}_M(\mathrm{sinc})$, then $|\widetilde{\Gamma}_{\mathrm{sinc}\,\upsilon}\left(\frac{u}{b}\right)| = \frac{1}{\sqrt{2\pi}}$, in which case a perfect signal recovery is possible. Otherwise, our derived results demonstrate that the ideal bandlimited signal approximation $\mathcal{P}_{\mathcal{V}_M}f(t)$ can not be obtained exactly from the samples, unless the input signal $f(t)$ is itself bandlimited in the LCT domain. One may be tempted to conclude that the proposed algorithm can be viewed as a special type of deconvolution technique. The major difference between our method and ordinary deconvolution methods is that our approach combines the task of signal approximation and correction for sensor distortions. Moreover, the resulting algorithm is digital although the problem is initially formulated in the continuous signal domain. In this sense, the proposed algorithm provides a proper discretization for an analog deconvolution problem.

Other potential applications of the derived results can be found in the design of correction filters for improving the rendition of digital images on a display device (e.g., video monitor, film recorder, etc.). We will make the standard assumption that the digital images were acquired using a measurement procedure that conforms with Shannon's sampling theorem of the LCT and consider the case of a display device that use piecewise constant interpolation. The corresponding basis functions $\upsilon(x,y) = \mathrm{sinc}(x)\,\mathrm{sinc}(y)$ and $\phi(x,y) = \beta^0(x)\beta^0(y)$ are separable in the x and y coordinates (x and y represent the horizontal and vertical spatial dimension, respectively). Then, we can use our derived results to obtain a separable correction filter that is implemented by successive one-dimensional processing along the rows and the columns of the image. The DTFT of the resulting one-dimensional digital filter specified by Theorem 1 is given by

$$\widetilde{H}\left(\tfrac{u}{b}\right) = \frac{1}{\frac{1}{\sqrt{2\pi}}\,\mathrm{sinc}(\frac{u}{2\pi b})}, \quad u\in\mathbb{I} \tag{112}$$

which is plotted in Fig. 6(a). The effect of this filter is to enhance higher spatial LCT-frequencies, which corresponds to a special form of image sharpening [40]. In order to verify the proposed image sharpening technique, we have used the cameraman image sequence. The original image is shown in Fig. 6(b), and Fig. 6(c) illustrates the corresponding sharpened

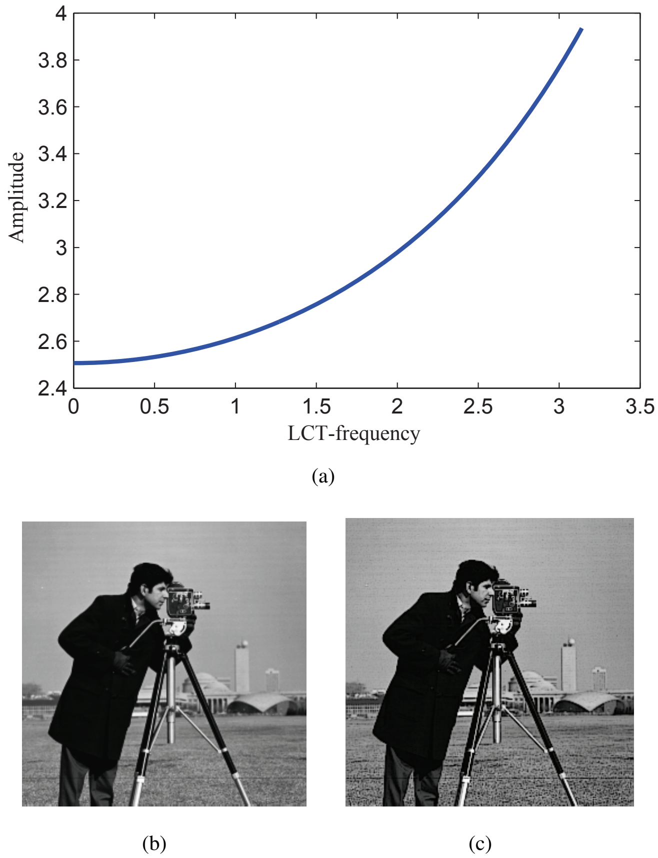

Fig. 6. Image sharpening based on the proposed method: (a) LCT-frequency response of the one-dimensional digital correction filter for a piecewise constant signal reconstruction; (b) original image; (c) sharpened image.

image. It can be observed that fine details are well sharpened using the proposed image sharpening method.

VI. Conclusion

In this paper, we have proposed a generalized sampling scheme associated with the LCT, which allows for arbitrary measurement and approximation spaces. The approximation space is usually determined by the digital to analog conversion algorithm. The only addition to the standard sampling scheme is a digital correction filter in the LCT domain. No special constraint, such as bandlimitedness, is imposed on the input signal. Our only requirement is that the reconstructed signal is indistinguishable from the input in the sense that it yields the exact same measurements. Therefore, the proposed sampling scheme may be applicable to most practical situations. Moreover, a spectral coherence function that measures the "similarity" between the measurement and approximation spaces is introduced, and a relative performance bound for the comparison with the least squares solution is derived. Numerical results and several potential applications of the derived results are also presented.

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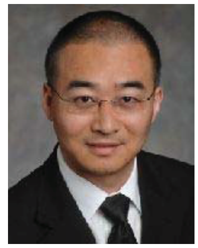

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